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Complexity of flow shop scheduling problems with transportation constraints

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Abstract

In most manufacturing and distribution systems, semi-finished jobs are transferred from one processing facility to another by transporters such as Automated Guided Vehicles, robots and conveyors, and finished jobs are delivered to warehouses or customers by vehicles such as trucks.

This paper investigates two-machine flow shop scheduling problems taking transportation into account. The finished jobs are transferred from the processing facility and delivered to customers by truck. Both transportation capacity and transportation times are explicitly taken into account in these models. We study the class of flow shop problems by analysing their complexity. For the makespan objective function, we prove that this problem is strongly NP-hard when the capacity of a truck is limited to two or three parts with an unlimited buffer at the output of the each machine. This problem with additional constraints, such as blocking, is also proven to be strongly NP-hard.

Keywords: Flexible manufacturing systems; Flow shop; Scheduling; Complexity

1. Introduction

A Flexible Manufacturing System (FMS) is an integrated system composed of automated material handling devices and of numerically controlled machines that can process a variety of part types [12]. This system has been introduced to increase flexibility by overcoming the traditional hypotheses assuming that either there is an infinite number of transporters for job-delivering or that jobs are instantaneously transported from one location to another with no transportation time taken into account.

In fact, in most manufacturing and distribution systems, semi-finished jobs are transferred from one processing facility to another by transporters such as Automated Guided Vehicles (AGVs), robots and conveyors, and finished jobs are delivered to warehouses or customers by vehicles such as trucks.

One important issue encountered when managing an FMS is the scheduling problems. Scheduling problems arise when a set of jobs is processed by a set of resources according to their manufacturing

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processes. Unfortunately, most production scheduling problems are very complex and have been proven to be NP-hard.

This paper addresses a two-machine flow shop scheduling problem with a material handling system. In this system, n jobs will be processed on the two machines M_1 and M_2. The finished jobs are transferred from the processing facility and delivered to customers by a truck. The capacity of the truck is limited to a fixed "c" parts (the truck can transport at most c parts at a time). The processing time of a job T_j $(j = 1, \dots, n)$ on the machine M_1 and the machine M_2 are respectively denoted by p_{1j} and p_{2j}. These two machines are discrete processors (processing at most one job at a time).

In [11], the authors investigate machine scheduling models that take constraints on both the transportation capacity and transportation times into account. They consider two types of transportation. The first one, denoted "type-1" transportation, is intermediate transportation in a flow shop where semi-finished jobs are transferred from one machine to the next. The second one, denoted "type-2" transportation, is the transportation necessary to deliver finished jobs to the customers. Some results of computational complexity are given and "open" problems are underlined.

This work has led to the study of the complexity of flow shop scheduling problems in which the capacity of the truck is limited to two or three jobs.

The second machine M_2 can be considered as a transporter such as a "robot". So, the processing facility will be called Single Manufacturing System (SMS), as it was noted in [12]. Several authors have also investigated the scheduling problem in this type of FMS. For instance, [2] called it Flexible Machining Cell (FMC), and Kusiak [9] denoted it Flexible Manufacturing Module (FMM) (see Fig. 1).

This problem can be also viewed as a workshop, where jobs can be gathered into batches depending on the capacity of resources: two discrete machines composed the processing facility and the truck represents the batch machine with capacity c. In the literature, we find two types of batch processing machines [5,13]: a max-batch machine that simultaneously processes a subset of parts, the processing time of a batch is equal to the largest processing time of parts in this batch and a sum-batch machine that can sequentially process several parts, the processing time of a batch is equal to the sum of the processing times of all parts in the batch. In [1] the authors consider a situation in which the manufacturing system is equipped with batch and discrete processors. They analyse the complexity status for the two-machine scheduling problems.

This paper is organised as follows: In Section 2 we describe the used notations. Then, we study the two-machine flow shop type-2 transportation by classifying their computational complexity under the following constraints: a buffer storage capacity, a transportation time and a truck capacity. In Section 3, we study the complexity of the two-machine flow shop problem taking transportation into account and with sufficient buffer between machines. The capacity of the truck is limited to two parts. We show that this problem is strongly NP-hard. The case in which there is no buffer space between the two machines can be deduced

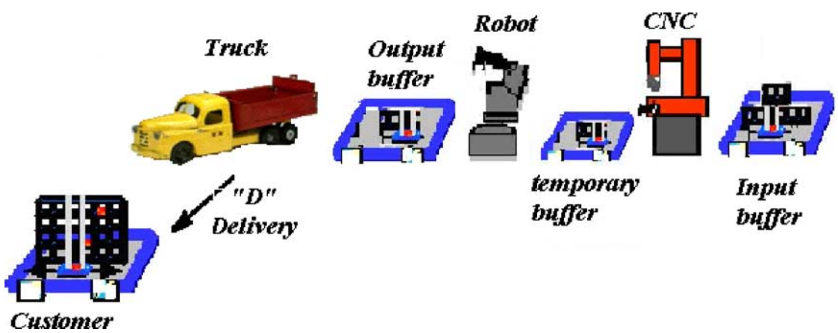

Fig. 1. Single Manufacturing System with truck as transporter of parts to customer.

from the previous result. In Section 4, we study the case in which the capacity of the truck is limited to three parts and the transportation times are a constant value. We show that this problem is strongly NP-hard. Similarly, the case in which there is no buffer space between the two machines can be deduced from the previous result.

2. Problem description

In most manufacturing distribution systems, finished jobs are transferred from a processing facility and delivered to customers or warehouses by truck. This paper investigates the flow shop type-2 transportation.

The flow shop type-2 transportation can be described as follows: consider two machines M_1 and M_2 and n jobs. Each job T_j consists of two operations O_{1j} and O_{2j} which have to be processed in the order $O_{1j} \rightarrow O_{2j}$, $j = 1, \ldots, n$, without preemption on each machine. The processing times of the two operations are respectively given by p_{1j} and p_{2j} and operation O_{2j} can only start after O_{1j} has been finished. At time t one machine can only process one operation. The finished jobs must be transported and delivered to one and only one customer using the truck. This vehicle has a capacity "c". It means that at time t the truck can transport at most c jobs.

By using the general notation for scheduling problems, introduced by Graham et al. [6] and according to the notation of Lee and Chen [11] the problem of minimising the makespan for the two-machine flow shop type-2 transportation problem is denoted by $F2 \rightarrow D \mid v, c_i \mid C_{\max}$. The notation, $F2 \rightarrow D$ means that the jobs are processed firstly on two flow shop machines and then delivered to the customer. Let v denotes the number of identical trucks and c_i the capacity of the truck i. One truck ($v = 1$) is considered in this paper for $c \in \{2, 3\}$ and at the time $t = 0$ the truck is located at the output of the processing facility. The makespan is given by the delivered time of the last job plus the necessary time to come back to the processing facility.

The classical two-machine flow shop problem with unlimited intermediate buffer capacity and infinite speed of vehicles is polynomial [8]. In [11] the authors show that when constraints on both transportation capacity and transportation times are explicitly considered, this problem becomes strongly NP-hard. Table 1 summarises results for the computational complexity for the two-machine flow shop type-2 transportation and an unlimited intermediate buffer capacity.

Table 1
Complexity status of two-machine flow shop with type-2 transportation

Problem	Complexity
$F2 \rightarrow D \mid v = 1, c = 1 \mid C_{\max}$	Strongly NP-hard
$F2 \rightarrow D \mid v = 1, c = 2 \mid C_{\max}$	Open
$F2 \rightarrow D \mid v = 1, c = 3 \mid C_{\max}$	Open
$F2 \rightarrow D \mid v = 1, c \geqslant 4, \text{fixed } c \mid C_{\max}$	Strongly NP-hard
$F2 \rightarrow D \mid v = 1, c = \frac{n}{2} \mid C_{\max}$	NP-hard
$F2 \rightarrow D \mid v = 1, c \geqslant n - k, \text{fixed } k \mid C_{\max}$	Polynomial

Another case will be studied when the intermediate buffer capacity is equal to zero. In this system, we must consider the blocking constraints. The two-machine flow shop problem with blocking constraints is equivalent to the no-wait two-machine flow shop problem [7]. Gilmore and Gomory [4] have proved that minimising the makespane for the no-wait two-machine flow shop, denoted $F2 \mid \text{no-wait} \mid C_{\max}$ is polynomial. This problem is a special case of the Travelling Salesman Problem (TSP). Even if the TSP is strongly NP-hard, many special cases can be easily and efficiently solved [10].

When the truck is considered as a machine and has a capacity one, the problem $F2 \rightarrow D \mid v = 1, c = 1, \text{no-wait} \mid C_{\max}$ is equivalent to $F3 \mid \text{no-wait} \mid C_{\max}$, that is generally strongly NP-hard [14]. However, if the transportation times are equal to constant h (t_j, $j = 1, \ldots, n$) then this problem becomes polynomial [15]. Van der Veen and Van Dal [15] study the case of fixed processing times on $m - 2$ machines and obtain a polynomial-time algorithm for this problem.

3. Capacity of truck limited to two parts

In this section, the jobs are executed by the two machines composing the processing facility. Then the only truck transports the finished jobs. We first prove that the scheduling problem $F2 \to D\,|\,v=1, c=2\,|\,C_{\max}$ with an unlimited buffer area between machines is strongly NP-hard. The optimisation problem known as *NUMERICAL MATCHING WITH TARGET SUMS*, that is strongly NP-hard [3], is used to obtain the proof by solving the decision problem of $F2 \to D\,|\,v=1, c=2\,|\,C_{\max}$ using a polynomial reduction.

Problems with additional constraints, such as blocking constraint when there is no buffer area between the two machines, are also proved to be strongly NP-hard.

3.1. Flow shop with unlimited buffer

As indicated in the title of this sub-section, there is an unlimited buffer area both between the two machines and at the output of the processing facility. Let t_i be the necessary time to transport a job T_i $(i = 1, \dots, n)$ from the output system to the customer and to come back.

Theorem 1. *The problem* $F2 \to D\,|\,v=1, c=2\,|\,C_{\max}$ *is strongly NP-hard.*

Proof. We will show that the strongly NP-hard problem *NUMERICAL MATCHING WITH TARGET SUMS* denoted "NMTS" [3] is reducible to the decision problem of $F2 \to D\,|\,v=1, c=2\,|\,C_{\max}$ denoted Π.

NMTS: Instance: Let $A = \{a_i/i = 1, \dots, n\}$, $B = \{b_i/i = 1, \dots, n\}$ and $E = \{E_i/i = 1, \dots n\}$ be sets of positive integers, such that $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} E_i$.

Question: Does there exist a partition of $A \cup B$ into n disjoints pairs $I_1, I_2, \dots, I_n$, each pair I_k containing exactly one element from A and B, such that $a_{I_k} + b_{I_k} = E_k$, $k = 1, \dots, n$?

We show that a permutation π for the constructed instance with $C^{\pi}_{\max} \leqslant y$ exists if and only if NMTS has a solution.

Given an arbitrary instance I of NMTS, firstly we modify I to get an instance I' of NMTS as follows: Let $w = \max\{a_1, \dots, a_n, b_1, \dots, b_n\}$ and let $A' = \{a_1, \dots, a_n\}$, $B' = \{b_1 + 3w, \dots, b_n + 3w\}$ and $E' = \{E_1 + 3w, \dots, E_n + 3w\}$. Obviously, the instance I' of the problem NMTS has a solution, if and only if, the instance I has a solution.

We consider an arbitrary instance I' of NMTS and we construct one equivalent instance of the problem Π as follows:

Let us define an instance of $2n^2 + 2(n+1)$ jobs split into three subsets:

- A-jobs are denoted $A_{i,j}$ $(i, j = 1, \dots, n)$.
- B-jobs are denoted $B_{i,j}$ $(i, j = 1, \dots, n)$.
- U-jobs are denoted $U_{i,j}$ $(i = 1, \dots, n+1,\ j = 1, 2)$.

The processing times of the $2n^2 + 2(n+1)$ jobs on the two machines and also the transportation times are given in Table 2.

We demonstrate that the problem NMTS has a solution if and only if Π has solution π in which the makespan $C^{\pi}_{\max} \leqslant (n+1)\sum_{j=1}^{n} E_j$.

We assume that NMTS has a solution and let $N_1, \dots, N_n$ be the subsets of the indexes of jobs with: $a_l + b_j = E_k$, $l \in N_k$ and $j \in N_k$. So there exists a permutation π with $C^{\pi}_{\max} = (n+1)\sum_{i=1}^{n} E_i$.

Table 2
Jobs processing times

		$p_{1,j}$	$p_{2,j}$	$p_{3,j}$
A-jobs	$A_{i,j}$, $i, j = 1, \dots, n$	a_j	0	E_j
B-jobs	$B_{i,j}$, $i, j = 1, \dots, n$	b_j	0	E_j
U-jobs	$U_{1,1}$	0	0	E_1
	$U_{1,2}$	0	0	E_1
	$U_{i,j}$, $j = 1, i = 2, \dots, n$	E_{i-1}	0	E_i
	$U_{i,j}$, $j = 2, i = 2, \dots, n$	0	0	E_i
	$U_{n+1,1}$	E_n	0	0
	$U_{n+1,2}$	0	0	0

Let us batch the subset of two jobs that are simultaneously transported by the truck. We identify two classes of batch that are defined as follows:

C_1: batches of type $\{U_{k,1}, U_{k,2}\}$ for $k = 1, \dots, n+1$ and their transportation time is E_k, except for the batch $\{U_{n+1,1}, U_{n+1,2}\}$ for which the transportation time is equal to 0;

C_2: batches of type $\{A_{k,l}, B_{k,j}\}$ for $k = 1, \dots, n$, with $l \in N_k$, and $j \in N_k$ and their transportation time is E_k.

The class C_2 contains n batches of type $\{A_{k,l}, B_{k,j}\}$ for all k, $k = 1, \dots, n$.

The jobs in the same batch of the two classes C_1 and C_2 are sequenced in the same order on the two machines and are transported at the same time by the truck.

The first batch $\{U_{1,1}, U_{1,2}\}$ belonging to C_1 is sequenced on the first position on π. Let r_k be the position of the batch $\{U_{k,1}, U_{k,2}\}$ in π, so r_k is equal to $r_{k-1} + 2(n - k + 2)$.

We denote by $G_{k,i}$ the batches of C_2 such as $G_{k,i} = \{A_{k,l}, B_{k,j}\}$ with $l \in N_i$, $j \in N_i$ and has a transportation time equal to E_i $(i, k = 1, \dots, n)$. The position $r_{k,i}$ of the batch $G_{k,i}$ in π is given by the following formula:

$$r_{k,i} = \begin{cases} r_k + 2(i-k) + 1 & \text{if } k \leqslant i, \\ r_i + 2(k-i) & \text{if } k > i, \end{cases}$$

where r_j $(j = 1, \dots, n)$ is the position of the jth batch of the class C_1.

With the previous construction of the sequence π we have $C^{\pi}_{\max} = (n+1)\sum_{j=1}^{n} E_j$.

The following example explains the sequencing of the jobs for the instance NMTS with $n = 4$. We define $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$ and $E = \{E_1, E_2, E_3, E_4\}$. Suppose that NMTS has the following solution: $N_1 = \{a_1, b_2 / a_1 + b_2 = E_1\}$, $N_2 = \{a_2, b_1 / a_2 + b_1 = E_2\}$, $N_3 = \{a_3, b_4 / a_3 + b_4 = E_3\}$, $N_4 = \{a_4, b_3 / a_4 + b_3 = E_4\}$. After the construction of classes C_1 and C_2, we have the batches construction as shown in Fig. 2.

The schedule π of batches is defined by Fig. 3 in which the power on each batch gives its position in the sequence π.

Fig. 4 illustrates the permutation π.

Obviously, the makespan of schedule π is equal to $5\sum_{i=1}^{4} E_i$.

Conversely, let us assume that the problem Π has a solution π where

$$C^{\pi}_{\max} \leqslant (n+1)\sum_{i=1}^{n} E_i.$$

The completion time of all the jobs on the first machine is given by

$$C_1: \begin{pmatrix} 0,0 \\ 0,0 \\ E_1,E_1 \end{pmatrix} \begin{pmatrix} E_1,0 \\ 0,0 \\ E_2,E_2 \end{pmatrix} \begin{pmatrix} E_2,0 \\ 0,0 \\ E_3,E_3 \end{pmatrix} \begin{pmatrix} E_3,0 \\ 0,0 \\ E_4,E_4 \end{pmatrix} \begin{pmatrix} E_4,0 \\ 0,0 \\ 0,0 \end{pmatrix}$$

$$C_2: \begin{matrix} \begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_1,E_1 \end{pmatrix} & \begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_1,E_1 \end{pmatrix} & \begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_1,E_1 \end{pmatrix} & \begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_1,E_1 \end{pmatrix} \\ \begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_2,E_2 \end{pmatrix} & \begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_2,E_2 \end{pmatrix} & \begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_2,E_2 \end{pmatrix} & \begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_2,E_2 \end{pmatrix} \\ \begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_3,E_3 \end{pmatrix} & \begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_3,E_3 \end{pmatrix} & \begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_3,E_3 \end{pmatrix} & \begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_3,E_3 \end{pmatrix} \\ \begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_4,E_4 \end{pmatrix} & \begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_4,E_4 \end{pmatrix} & \begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_4,E_4 \end{pmatrix} & \begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_4,E_4 \end{pmatrix} \end{matrix}$$

Fig. 2. Batches construction.

$$
\begin{array}{lllll}
C_1: & C_2: & & & \\
\begin{pmatrix} 0,0 \\ 0,0 \\ E_1,E_1 \end{pmatrix}^{01} &
\begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_1,E_1 \end{pmatrix}^{02} &
\begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_1,E_1 \end{pmatrix}^{04} &
\begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_1,E_1 \end{pmatrix}^{06} &
\begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_1,E_1 \end{pmatrix}^{08} \\
\begin{pmatrix} E_1,0 \\ 0,0 \\ E_2,E_2 \end{pmatrix}^{09} &
\begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_2,E_2 \end{pmatrix}^{03} &
\begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_2,E_2 \end{pmatrix}^{10} &
\begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_2,E_2 \end{pmatrix}^{12} &
\begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_2,E_2 \end{pmatrix}^{14} \\
\begin{pmatrix} E_2,0 \\ 0,0 \\ E_3,E_3 \end{pmatrix}^{15} &
\begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_3,E_3 \end{pmatrix}^{05} &
\begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_3,E_3 \end{pmatrix}^{11} &
\begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_3,E_3 \end{pmatrix}^{16} &
\begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_3,E_3 \end{pmatrix}^{18} \\
\begin{pmatrix} E_3,0 \\ 0,0 \\ E_4,E_4 \end{pmatrix}^{19} &
\begin{pmatrix} a_1,b_2 \\ 0,0 \\ E_4,E_4 \end{pmatrix}^{07} &
\begin{pmatrix} a_2,b_1 \\ 0,0 \\ E_4,E_4 \end{pmatrix}^{13} &
\begin{pmatrix} a_3,b_4 \\ 0,0 \\ E_4,E_4 \end{pmatrix}^{17} &
\begin{pmatrix} a_4,b_3 \\ 0,0 \\ E_4,E_4 \end{pmatrix}^{20} \\
\begin{pmatrix} E_4,0 \\ 0,0 \\ 0,0 \end{pmatrix}^{21} & & & &
\end{array}
$$

Fig. 3. Sequence π.

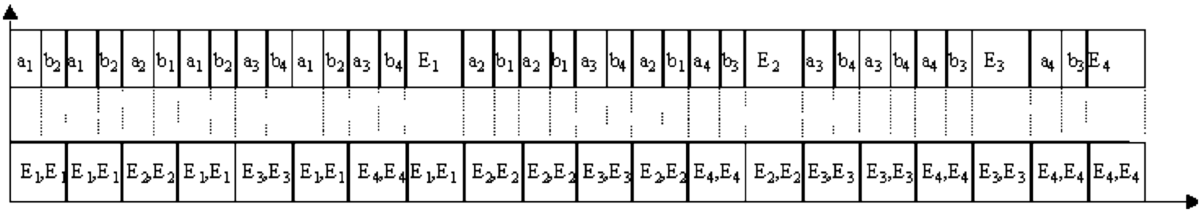

Fig. 4. Gantt diagram to illustrate the permutation π.

$$n\sum_{j=1}^{n} a_j + n\sum_{j=1}^{n} b_j + \sum_{j=1}^{n} E_j = (n+1)\sum_{j=1}^{n} E_j,$$

thus there is no idle time on machine M_1. We assume that the jobs are grouped in v batches, where $v \geqslant n^2 + n + 1$, because the capacity of machine D is limited to two jobs. Let $P(B_k)$ be the transportation time of the batch B_k ($k = 1, \ldots, v$). For each batch B_k, we define $P'(B_k)$ with $P'(B_k) = 0$ if batch B_k contains only one job and $P'(B_k) = \min\{t_j / j \in B_k\}$ otherwise. As the capacity of machine D is limited to two jobs, we have

$$\sum_{k=1}^{v} P(B_k) + \sum_{k=1}^{v} P'(B_k) = \sum_{k=1}^{2n^2+2(n+1)} t_k = 2(n+1)\sum_{k=1}^{n} E_i$$

and as $C_{\max} \leqslant (n+1)\sum_{i=1}^{n} E_i$, we have

$$\sum_{k=1}^{v} P'(B_k) = (n+1)\sum_{i=1}^{n} E_i.$$

So we establish that:

1. each batch B_k contains exactly two jobs, i.e. $v = n^2 + n + 1$;
2. the batches are composed of the jobs having the same transportation time, i.e. if one batch contains job $A_{i,j}$ then it contains $A_{k,j}$ or $B_{l,j}$ or $U_{q,j}$.

We remark that the sum of the transportation time of the batches is equal to $(n+1)\sum_{i=1}^{n} E_i$, i.e. there is no idle time on machine D. So we establish that only the batch $\{U_{1,1}, U_{1,2}\}$ can be arranged in the first position. This first batch is composed of the jobs with processing times equal to 0 on M_1 or M_2. This batch is transported in the interval $t = 0$ and $t = E_1$. In the interval ($t \in [0, E_1]$) machine M_1

consecutively processes two jobs. Then, at time $t = E_1$, they are immediately transported by the truck.

The second batch that will be transported by the truck and for which the earliest transportation time is equal to E_1, is formed by one element of A-jobs and one element of B-jobs and the sum of their processing times on M_1 is equal to E_1. We can assume that the transportation time of each of the last two jobs is equal to E_2. In fact, these two jobs form the first subset N_1 of NMTS.

Using the previous argumentation, we show that the remaining batches contain exactly two elements. Thus, we obtain schedule π with $C^{\pi}_{\max} = (n+1)\sum_{i=1}^{n} E_i$. Its structure is similar to the sequence obtained in the first part of this demonstration. □

3.2. Flow shop with limited buffer area

Now, we suppose that there is no buffer area between the machines. The absence of intermediate buffers between machines induces the blocking of jobs when the downstream machines are busy. Of course, one job cannot leave a machine until the next machine is free. In [7] the authors use "block$(i, i+1)$" to indicate that there exists blocking constraint between machines i and $i+1$. According to [7], this problem is denoted by $F2 \rightarrow D|v = 1, c = 2, \text{block}(1,2)|C_{\max}$.

Theorem 2. *The problem* $F2 \rightarrow D|v = 1, c = 2, \text{block}(1,2)|C_{\max}$ *is strongly NP-hard.*

Proof. With the same instance as given in the previous section, the strongly NP-hard problem NMTS is reducible to the decision problem of $F2 \rightarrow D|v = 1, c = 2, \text{block}(1,2)|C_{\max}$. It can be proven in the same way. Let us not forget that machine M_2 can be considered as a one part temporary buffer. □

4. Capacity of truck limited to three parts

In this section, we study the $F2 \rightarrow D|v = 1, c = 3|C_{\max}$. This system is composed of two machines M_1 and M_2 and n jobs. Two cases are treated in this section: $F2$ with unlimited buffer capacity and $F2$ without buffer space between the two first machines. But at the output of the processing facility, there is a buffer with a capacity $k \geqslant 1$.

4.1. Flow shop with unlimited buffer space

In this system there is an unlimited intermediary storage space capacity. Let t_j be the necessary time to transport a job T_j $(j = 1, \ldots, n)$ from the output system to the customer and to come back. Initially, the vehicle is located at the output of M_2. The jobs will be processed on machine M_1 and then on M_2. Then they will be transported to the customer by the truck that has a capacity equal to 3.

Theorem 3. *The problem* $F2 \rightarrow D|v = 1, c = 3, t_j = h|C_{\max}$ *is strongly NP-hard.*

Proof. We will show that the strongly NP-hard problem 3-PARTITION [3] is reducible to the decision problem of $F2 \rightarrow D|v = 1, c = 3,\ t_j = h|C_{\max}$ where t_j is a constant h for all jobs $j = 1, \ldots, n$.

3-PARTITION: Instance: Let $A = \{a_i/i = 1, \ldots, 3m\}$ be a set of positive integers, B a positive integer such that

$$\sum_{i=1}^{3m} a_i = mB \quad \text{and} \quad \frac{B}{4} < a_i < \frac{B}{2} \quad \text{for } i = 1, \ldots, 3m.$$

Question: Does there exist a partition of A into m disjoints sets $I_1, I_2, \ldots, I_m$ of indices such that $|I_j| = 3$ and $\sum_{i \in I_j} a_i = B$ for $j = 1, \ldots, m$?

We show that a permutation π for the constructed instance with $C^{\pi}_{\max} \leqslant y$ exists if and only if 3-PARTITION has a solution.

Given an arbitrary instance of 3-PARTITION, we construct the following instance of the flow shop problem $F2 \rightarrow D|v = 1, c = 3, t_j = h|C_{\max}$ with $n = 3m + 3$ jobs. The processing times of the jobs are given in Table 3.

The transportation time h is set to $h = B$ and we ask for a permutation π with

$$C_{\max} \leqslant y = (m+1)B.$$

Table 3
Jobs processing times

	$p_{1,j}$	$p_{2,j}$	$t_j = p_{3j}$
Jobs T_j, $j = 1, \dots, 3m$	a_j	0	B
Jobs T_j, $j = \{3m+1, 3m+2, 3m+3\}$	0	0	B

We assume that $I_1, I_2, \dots, I_m$ is a solution of 3-PARTITION. Then we define a permutation π as showed in Fig. 5.

Obviously, this permutation π fulfils $C^{\pi}_{\max} \leqslant y = (m+1)B$. In fact I_i $(i = 1, \dots, m)$ represents the jobs that will be simultaneously executed by the last machine D where $I_i = \{T_j / j = 1, \dots, 3m\}$, $i = 1, \dots, m$, and $I_{m+1} = \{T_{3m+1}, T_{3m+2}, T_{3m+3}\}$. Permutation π is defined by $(I_{m+1}, I_1, \dots, I_m)$.

Conversely, let us consider that the problem $F2 \rightarrow D | v = 1, c = 3, t_j = h | C_{\max}$ has a solution in which $C^{\pi}_{\max} \leqslant y$.

The objective value of one permutation $s \in S$ (S is a set of feasible solutions) is given by the value of a critical path. So for n jobs ($n = 3m + 3$) the makespan is greater or equal to $(m+1)B$. Thus, for a permutation π with $C^{\pi}_{\max} = (m+1)B$, we may conclude that:

1. truck D transports jobs in the interval $[0, (m+1)B]$ without idle time;
2. truck D starts at each time $t = k * B (k = 0, \dots, m+1)$ with exactly three jobs;
3. jobs $T_{3m+1}, T_{3m+2}, T_{3m+3}$ are processed at the first position, since $p_{1j} > 0$ for $j \in \{3m+1, 3m+2, 3m+3\}$ and therefore the truck cannot deliver a real part before the three first ones are ready;
4. machine M_1 processes the jobs in the interval $[0, mB]$ without idle time.

For the permutation π with $C^{\pi}_{\max} = (m+1)B$, the truck can move at most $m+1$ times, and the necessary time to transport all the jobs in π is equal to $(m+1)B$. It means that there is no idle time on the machine D. Otherwise, the makespan would be greater than $(m+1)B$, which contradicts the conclusion 1.

In the following we will show that the truck simultaneously transports three and only three jobs. We have $C^{\pi}_{\max} = (m+1)B$. So the truck can transport at most $m+1$ groups of jobs. As the capacity of the truck is equal to three and as the total number of the jobs is equal to $3m+3$ the truck must transport exactly three jobs at each of its move. Otherwise, the $C^{\pi}_{\max}$ would be greater than $(m+1)B$. So each group of three jobs forms a set denoted I_i $(i = 1, \dots, m+1)$.

In the following, we will show that jobs T_{3m+1}, T_{3m+2} and T_{3m+3} are processed at the first position.

Let $p_{k,I_i}(j)$ be the processing time of the jth job $(j = 1, 2, 3)$ of the set I_i which must be executed on machine M_k. And let $I_1, I_2, \dots, I_{m+1}$ be the ordered subsets given in the sequence π. A lower bound of $C_{\max}$ can be calculated as follows:

Considering subset I_k, the lower bound can be estimated by adding three elements: first, the sum of the processing times of the $(k-1)$th first subsets formed by their belonging jobs, secondly the processing time on M_1 of each job forming I_k, added to the transportation times of the three jobs forming I_k, and thirdly the transportation time of the jobs forming the last $(m+1-k)$th subsets.

Thus, the lower bound is given by the following formula:

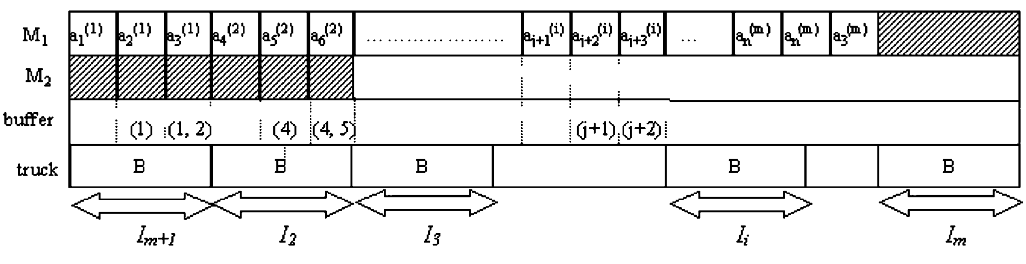

Fig. 5. Gantt diagram to illustrate the permutation π.

$$\sum_{i=1}^{k-1}[p_{1,I_i(1)}+p_{1,I_i(2)}+p_{1,I_i(3)}]+p_{1,I_k(1)}+p_{1,I_k(2)}$$
$$+p_{1,I_k(3)}+(m-k+2)B\leqslant(m+1)B$$
$$\Longleftrightarrow\sum_{i=1}^{k}[p_{1,I_i(1)}+p_{1,I_i(2)}+p_{1,I_i(3)}]$$
$$+(m-k+2)B\leqslant(m+1)B$$
$$\Longleftrightarrow\sum_{i=1}^{k}[p_{1,I_i(1)}+p_{1,I_i(2)}+p_{1,I_i(3)}]\leqslant(k-1)B.$$

For $k=1$, we have

$$p_{1,I_1(1)}+p_{1,I_1(2)}+p_{1,I_1(3)}\leqslant 0.$$

It means that the jobs T_{3m+1}, T_{3m+2} and T_{3m+3} form the first subset I_1 in π and are processed first by M_1.

For $k=2$, we have

$$p_{1,I_1(1)}+p_{1,I_1(2)}+p_{1,I_1(3)}+p_{1,I_2(1)}+p_{1,I2(2)}+p_{1,I_2(3)}\leqslant B\ \Rightarrow\ p_{1,I_2(1)}+p_{1,I_2(2)}+p_{1,I2(3)}\leqslant B.$$

For $k=3$, we have

$$p_{1,I_2(1)}+p_{1,I_2(2)}+p_{1,I_2(3)}+p_{1,I_3(1)}+p_{1,I_3(2)}+p_{1,I_2(3)}\leqslant 2B$$

as $p_{1,I_2(1)}+p_{1,I_2(2)}+p_{1,I_2(3)}\leqslant B$
then $p_{1,I_3(1)}+p_{1,I_3(2)}+p_{1,I_3(3)}\leqslant B$.

So for each j ($j=2,\dots,m+1$) we have
$p_{1,I_j(1)}+p_{1,I_j(2)}+p_{1,I_j(3)}\leqslant B$.

But $\sum_{i=2}^{m+1}[p_{1,I_i(1)}+p_{1,I_i(2)}+p_{1,I_i(3)}]=mB$.
So for $j=2,\dots,m+1$ we have
$p_{1,I_j(1)}+p_{1,I_j(2)}+p_{1,I_j(3)}=B$.

Analogously, we show that the remaining sets $I_2,\dots,I_{m+1}$ contain three elements and fulfil $\sum_{k=1}^{2}p_{1,I_j(k)}=B$ for $j=2,\dots,m+1$. Thus, $I_2,\dots,I_{m+1}$ define a solution of 3-PARTITION. □

Corollary 1. *The problem* $F2\rightarrow D|v=1,c=3|C_{\max}$ *is strongly NP-hard.*

4.2. Flow shop with limited buffer space

In this sub-section, we consider another case of a two-machine flow shop scheduling problem, called type-2 transportation. A set of n jobs has to be processed on two machines M_1 and M_2. There is no storage space between machines M_1 and M_2. But, at the output of M_2, there is a buffer with a capacity $k\geqslant 1$. The parts will be processed on machine M_1 and then on M_2 in blocking condition (absence of a buffer between M_1 and M_2). When M_2 finishes the processing of the job, it is put into its intermediary buffer. When the truck is available, it picks up the part from the buffer to deliver it to the customer. So this problem is denoted by $F2\rightarrow D|v=1,c=3,t_j=h,\text{block}(1,2)|C_{\max}$.

Theorem 4. *Problem* $F2\rightarrow D|v=1,c=3,t_j=h,\text{block}(1,2)|C_{\max}$ *is strongly NP-hard.*

Proof. With the same instance as given in Section 4.1, the strongly NP-hard problem 3-PARTITION is reducible to the decision problem of $F2\rightarrow D|v=1,c=3,t_j=h,\text{block}(1,2)|C_{\max}$. It can be proven in the same way. Let us remind that machine M_2 could be considered as a unit temporary buffer. □

Corollary 2. *The problem* $F2\rightarrow D|v=1,c=3,\text{block}(1,2)|C_{\max}$ *is strongly NP-hard.*

5. Conclusions and perspectives

This paper investigates flow shop scheduling models that explicitly consider constraints on both transportation and buffer capacities. The finished jobs are transferred from the processing facility and delivered to one and only one customer or warehouse by a vehicle such as a truck. This vehicle has a capacity "c". It means that at time t the truck can transport at most c objects. The considered objective function is the makespan.

New complexity results are derived for special cases in which the capacity of the truck is limited to two jobs:

- with an unlimited buffer between machines, the problem is strongly NP-hard;
- with no buffer between the two machines, the problem is strongly NP-hard.

If the capacity of the truck is limited to three jobs and the transportation time is a constant value:

- with an unlimited buffer between machines, the problem is strongly NP-hard;
- with no buffer between the two machines, the problem is strongly NP-hard.

Many interesting problems remain for future exploration. Various polynomially solvable cases need to be identified, and when the problems are proved to be NP-hard, heuristic methods must be used. More realistic models need to be studied, including flow shop type-2 transportation involving multiple customers such that vehicle routing decisions would have to be addressed as well.

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