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Algorithmic analysis of the *Geo*/*Geo*/*c* retrial queue

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Abstract

In this paper, we consider a discrete-time queue of $Geo/Geo/c$ type with geometric repeated attempts. It is known that its continuous counterpart, namely the $M/M/c$ queue with exponential retrials, is analytically intractable due to the spatial heterogeneity of the underlying Markov chain, caused from the retrial feature. In discrete-time, the occurrence of multiple events at each slot increases the complexity of the model and raises further computational difficulties. We propose several algorithmic procedures for the efficient computation of the main performance measures of this system. More specifically, we investigate the stationary distribution of the system state, the busy period and the waiting time. Several numerical examples illustrate the analysis.

Keywords: Queueing; $Geo/Geo/c$ queue; Retrials; Stationary distribution; Busy period; Waiting time; Matrix-analytic methods

1. Introduction

In the various queueing frameworks, the customers that find all servers busy upon their arrival may abandon the system for ever (loss systems), wait in the system until a server becomes available (waiting systems) or repeat their request later (retrial systems). Most papers in early queueing theory have been devoted to loss and waiting systems because they allow in many cases exact analytic results. However, we cannot neglect the effect of repeated requests in most real applications in the fields of telephony, telecommunication systems, etc. This justifies the recent increase of the interest in retrial queues (see e.g. Falin and Templeton, 1997; Artalejo, 1999). We have to note that the majority of retrial systems, even the Markovian ones, are not analytically tractable. The reason lies in the spatial heterogeneity of the underlying processes, caused from the retrial feature. Therefore, many papers are focused in the development of efficient algorithmic solutions.

With the advent of digital systems, there also exists an emerging need for studying discrete-time queues. For a comprehensive survey of fundamental results in discrete-time queueing theory, the reader is referred to the

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books of Bruneel and Kim (1993), Takagi (1993) and Woodward (1994) and to the review article of Chaudhry (2000). This tendency has also influenced the research in the retrial literature. The basic discrete-time single server retrial model was studied in Yang and Li (1995). Subsequently, many authors have investigated a variety of discrete-time single server retrial systems, see e.g. Choi and Kim (1997), Li and Yang (1998, 1999), Takahashi et al. (1999), Atencia and Moreno (2004, 2006) and Artalejo et al. (2005).

The study of discrete-time multiserver retrial systems is a new endeavour. Recently, Artalejo and Lopez-Herrero (2005) presented a simulation study of a discrete-time multiserver retrial queue with finite population. In the present paper, we investigate a $Geo/Geo/c$ queue with geometric retrials. This model is the discrete-time analogue of the $M/M/c$ retrial queue (see Chapter 2 in Falin and Templeton (1997)).

The $M/M/c$ retrial queue is known to be analytically intractable and the analysis of its queue length, busy period and waiting time distributions is carried out using several approximation ideas (see, respectively, Artalejo and Pozo (2002), Artalejo et al. (2007) and Artalejo and Gómez-Corral (2005)). Indeed, the $M/M/c$ retrial queue is described by a continuous-time non-homogeneous QBD process. In the analysis of the $Geo/Geo/c$ retrial queue the occurrence of multiple events within a slot introduces additional difficulties. In particular, there exists a large variety of directly accessible states for each state. This complicates the matrix structure of the model and the $Geo/Geo/c$ retrial queue is reduced to a discrete-time non-homogeneous $GI/M/1$ type process. Hence, it seems promising to apply matrix-analytic methods, as it has been effectively done in a variety of retrial systems (see e.g. the detailed bibliography in Gómez-Corral (2006)).

Although the efficient computational analysis of non-homogeneous $GI/M/1$ type processes remains still an open problem, it is possible to exploit the matrix structure and sparsity of the blocks to proceed algorithmically. There are several alternatives for the efficient solution of the problems that appear in several places (e.g. block Gaussian elimination or censoring of Markov chains for stationary probabilities, truncation or not of the state space for obtaining the busy period characteristics, etc.). We discuss these alternatives in each case but, for the sake of brevity, we present in detail only those methods that we use for obtaining the numerical results.

The paper is organized as follows. In Section 2, we introduce some notation, describe the model and comment on its matrix structure. In Section 3, we derive the positive recurrence condition for the model and give two methods for approximating its stationary distribution, using truncation and generalized truncation. In Section 4, we proceed to the analysis of the busy period. We develop an exact algorithmic scheme for computing the probabilities associated with the length of a busy period and some approximate algorithms for the corresponding busy period moments. In Section 5, we study the waiting time distribution under a random order discipline for the customers in the retrial orbit. Finally, in Section 6, the various algorithms are illustrated with numerical results.

2. Model description and preliminaries

We assume that the time axis has been divided into equal intervals of unit length called slots. In the $Geo/Geo/c$ retrial queue, primary customers arrive to the system according to a Bernoulli process with probability p. Equivalently, the arrival process is a discrete renewal process with geometric interarrival times with probability mass function (p.m.f.) $p\bar{p}^{k-1}$, for $k \geqslant 1$, where $\bar{p} = 1 - p$. Service is rendered by c identical servers with service times geometrically distributed with p.m.f. $q\bar{q}^{k-1}$, for $k \geqslant 1$, where $\bar{q} = 1 - q$. This means that for any busy server the probability that it will finish the undergoing service in the next slot is q. An arriving customer who finds a free server upon his arrival begins immediately to be served. Otherwise, he joins the retrial orbit and reapplies for service later. Customers in the retrial orbit behave independently of each other and retry with probability s at every time slot, i.e., their retrial times are independent geometric random variables with p.m.f. $s\bar{s}^{k-1}$, for $k \geqslant 1$, where $\bar{s} = 1 - s$.

We assume a generalized early arrival scheme (G-EAS) in which all queueing events occur around the slot boundaries, see Fig. 1. We suppose that, at a given slot boundary t, departures occur in (t^-, t), while primary arrivals and retrials occur in (t, t^+). This information for the ordering of the events determines completely the stationary distribution of the system state and the busy period characteristics. For obtaining the waiting time distribution, we must be even more specific about which customers are selected for service at a given slot, if we have a primary arrival and/or many successful retrials that exceed the number of free servers. We suppose that

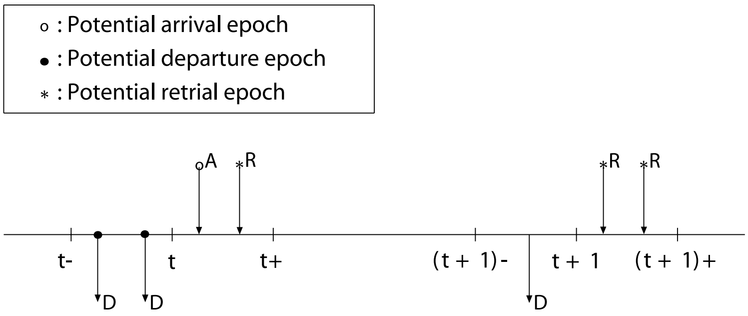

Fig. 1. Various epochs in G-EAS.

primary arrivals occur first so they have priority over retrials. Among retrying customers, the ones who occupy the servers are selected randomly, i.e., we assume the random order (RO) discipline.

The state of the system at time t^+ can be described by a process $\{(C_t, O_t);\ t \geqslant 0\}$, where C_t denotes the number of busy servers and O_t records the number of customers in the retrial orbit. The process $\{(C_t, O_t);\ t \geqslant 0\}$ is an irreducible Markov chain with state space $S = \{(i,j) | 0 \leqslant i \leqslant c,\ j \geqslant 0\}$ and one-step transition probabilities $P_{(i,j)(m,n)}$ given as in the following theorem.

Theorem 1. *The non-zero transition probabilities* $P_{(i,j)(m,n)}$ *of the process* $\{(C_t,\ O_t); t \geqslant 0\}$ *are as follows*:

(i) *If* $n = j + 1$ *and* $i = m = c$ *then*

$$P_{(c,j)(c,j+1)} = p\bar{q}^c. \tag{1}$$

(ii) *If* $\max\{0, j - c\} \leqslant n \leqslant j,\ j - n \leqslant m \leqslant c$ *and* $\max\{0, m + n - j - 1\} \leqslant i \leqslant c$, *then*

$$\begin{aligned} P_{(i,j)(m,n)} = &\left((1-\delta_{m,j-n+i+1})\bar{p}\binom{i}{i+j-m-n} q^{i+j-m-n}\bar{q}^{m+n-j} \right. \\ &\left. + (1-\delta_{m,j-n})p\binom{i}{i+j+1-m-n} q^{i+j+1-m-n}\bar{q}^{m+n-j-1} \right)\left(\binom{j}{j-n} s^{j-n}\bar{s}^{n} + \delta_{mc}\sum_{r=j-n+1}^{j}\binom{j}{r} s^{r}\bar{s}^{j-r} \right), \end{aligned} \tag{2}$$

where the symbol δ_{ab} *denotes Kronecker's function defined by*

$$\delta_{ab} = \begin{cases} 1, & \textit{if } a = b, \\ 0, & \textit{if } a \neq b. \end{cases}$$

Proof. Suppose that we denote by (i, j) and (m, n) the current state of the process and the next state to be visited, respectively. For any fixed j, the range for n is clearly from $\max\{0, j - c\}$ to $j + 1$. Indeed, the extreme case $n = j + 1$ occurs only when all servers are busy, there are no departures and one primary arrival occurs and joins the orbit. So we have $n = j + 1$ only in the case $i = m = c$ and it has the probability given in formula (1).

The other extreme case $n = j - c$ occurs if all the servers finish their services, no primary arrivals occur and at least c retrial customers repeat their request for service. This case implies that necessarily $m = c$ and accounts for the last factor of expression (2). In general, for having $m < c$ and $n \leqslant j$ customers in orbit in the next slot, we must have $r = j - n$ retrials which correspond to having $j - n$ successful trials in a binomial distribution with parameters j (number of trials) and s (probability of having a successful trial).

Let d be the number of departures and a the number of primary arrivals at the slot under consideration. Given the states (i, j) and (m, n), we have necessarily that $d - a = i + j - m - n$. This can happen either as

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$(d,a) = (i+j-m-n,0)$ or as $(d,a) = (i+j+1-m-n,1)$ which lead respectively to the two first terms in (2). Note that, for given values of j and n with $\max\{0, j-c\} \leqslant n \leqslant j$, the range for m is from $j-n$ to c. Indeed, we have at least $j-n$ retrial customers who will occupy servers at the next slot. Given j, n and m, the range for i is from $\max\{0, m+n-j-1\}$ to c since $0 \leqslant d \leqslant i$. Finally, we notice that $d = -1$ has no sense so transitions $(d,a) = (i+j-m-n,0)$ happen only for $i \neq m+n-j$–1; similarly, $d = i+1$ is also impossible so transitions $(d,a) = (i+j+1-m-n,1)$ happen only for $m \neq j-n$. This explains the use of Kronecker's factors in (2). □

Let $S_{(i,j)}$ be the set of one-step accessible states from the initial state (i,j). This set can be decomposed as follows:

$$S_{(i,j)} = \begin{cases} \cup_{k=0}^{i+1} D_j^k, & \text{if } 0 \leqslant i \leqslant c-1, \\ \cup_{k=0}^{c} D_j^k \cup \{(c, j+1)\}, & \text{if } i = c, \end{cases}$$

where $D_j^k = \{(k,j), (k+1, j-1), \ldots, (k+\min\{c-k, j\}, j-\min\{c-k, j\})\}$. The superscript k in D_j^k denotes the number of servers that remain occupied just after the departures and the primary arrival occurring in the next slot. The sets D_j^k are disjoint with cardinality $|D_j^k| = \min\{c-k, j\} + 1$. Then, it follows that the cardinality of $S_{(i,j)}$ is given by

$$|S_{(i,j)}| = \begin{cases} (i+2)(j+1), & \text{if } 0 \leqslant i \leqslant c-1, 0 \leqslant j \leqslant c-1-i, \\ (c-j+1)(j+1) + \frac{(j+c-i)(i+j+1-c)}{2} + \delta_{ic}, & \text{if } 1 \leqslant i \leqslant c, c-i \leqslant j \leqslant c-1, \\ \frac{(i+2)(2c+1-i)}{2} + \delta_{ic}, & \text{if } 0 \leqslant i \leqslant c\ j \geqslant c. \end{cases} \quad (3)$$

Eq. (3) shows that there exists a quite large number of immediately accessible states. Of course, this number increases as c grows. This is a crucial difference between the *Geo*/*Geo*/*c* retrial queue and the *M*/*M*/*c* retrial queue.

Because the number j of customers in orbit may increase at most by 1 or decrease at most by $\min\{c, j\}$ at each slot, the one-step transition probability matrix $\mathbf{P} = [P_{(i,j)(m,n)}]$ has a particular non-homogeneous *GI*/*M*/1 type structure. Let $l(j) = \{(0,j), \ldots, (c,j)\}$ denote the jth level of the process. Then, the matrix $\mathbf{P}$ can be partitioned in square blocks $\mathbf{A}_{jn}$ of dimension $c+1$ containing the transition probabilities from $l(j)$ to $l(n)$. We notice that $\mathbf{A}_{jn} = \mathbf{0}$, for $n < j-c$ and $n > j+1$. The blocks $\mathbf{A}_{j,j+1}$ have only one non-zero element, $P_{(c,j)(c,j+1)}$, while $\mathbf{A}_{jn}$, for $j-c \leqslant n \leqslant j$, have $c+1+n-j$ non-zero columns. These features of the matrix structure of the model are the basis for its algorithmic analysis.

3. The stationary distribution of the system state

In this section, we study the stationary behavior of the system state. As usual, we begin by investigating the system stability. It is well-known that the positive recurrence conditions for the retrial and standard *M*/*M*/*c* queues are identical. We expect to have the same situation for the *Geo*/*Geo*/*c* retrial queue because, as the number of retrying customers grows, the system approaches the standard *Geo*/*Geo*/*c* queue. This statement is confirmed in the following.

Theorem 2. *The Markov chain* $\{(C_t, O_t); t \geqslant 0\}$ *is positive recurrent if and only if*

$$p < cq. \quad (4)$$

Proof. We will use Foster's mean drift criterion for positive recurrence: An irreducible discrete-time Markov chain $\{X_t; t \geqslant 0\}$, with state space S, is positive recurrent if there exist a non-negative function f and a positive number ε such that the mean drifts $\gamma_s = E[f(X_{t+1}) | X_t = s] - f(s)$ are finite for all $s \in S$ and $\gamma_s < -\varepsilon$ for all states s, except perhaps a finite number.

We suppose that (4) holds and consider a linear function $f(i,j) = \alpha i + j$. Then, we have

$$\gamma_{(i,j)} = \sum_{(m,n)\in S_{(i,j)}} P_{(i,j)(m,n)}(\alpha m + n) - (\alpha i + j). \tag{5}$$

We are interested in finding the behavior of $\gamma_{(i,j)}$ as $j \to \infty$. To this end, we first establish the limiting behavior of $P_{(i,j)(m,n)}$ as $j \to \infty$. Since $j - n$ takes values on $\{0, \ldots, c\}$, for $n \leqslant j$, it follows that

$$\lim_{j\to\infty} \binom{j}{j-n} s^{j-n}\bar{s}^{n} = 0,$$

$$\lim_{j\to\infty} \sum_{r=j-n+1}^{j} \binom{j}{r} s^{r}\bar{s}^{j-r} = 1.$$

Hence, we see that formula (2) implies $P_{(i,j)(m,n)} \to 0$ as $j \to \infty$, for $0 \leqslant m \leqslant c-1$. In the case $m = c$, given the initial state (i,j) with $j \geqslant c$, the range of n is from $j-c$ to $j-c+i+1$, i.e., $n = j-c+k$, for $0 \leqslant k \leqslant i+1$. Thus, we have

$$\lim_{j\to\infty} P_{(i,j)(c,j-c+k)} = (1-\delta_{k,i+1})\bar{p}\binom{i}{i-k} q^{i-k}\bar{q}^{k} + (1-\delta_{k0})p\binom{i}{i+1-k} q^{i+1-k}\bar{q}^{k-1}, \quad 0 \leqslant k \leqslant i+1.$$

Note that $\sum_{k=0}^{i+1}\lim_{j\to\infty}P_{(i,j)(c,j-c+k)} = 1$, for each $(i,j) \in S$. Therefore, taking limit as $j \to \infty$ in (5) results in

$$\begin{aligned}\lim_{j\to\infty} \gamma_{(i,j)} &= \sum_{k=0}^{i+1} \lim_{j\to\infty} P_{(i,j)(c,j-c+k)}((\alpha c + j - c + k) - (\alpha i + j)) = \alpha c - c - \alpha i + \sum_{k=0}^{i+1} k \lim_{j\to\infty} P_{(i,j)(c,j-c+k)} \\ &= \alpha c - c - \alpha i + \bar{p}\sum_{k=0}^{i} k\binom{i}{i-k} q^{i-k}\,\bar{q}^{k} + p\sum_{k=1}^{i+1} k\binom{i}{i+1-k} q^{i+1-k}\,\bar{q}^{k-1} = \alpha c - c - \alpha i + i\bar{q} + p. \end{aligned} \tag{6}$$

For $0 \leqslant i \leqslant c-1$, we have from (6) that $\lim_{j\to\infty}\gamma_{(i,j)} \leqslant \alpha c - c - \alpha i + i + p$. Choose α such that $\alpha < 1 - p$. Then, we have $\gamma_{(i,j)} < -\varepsilon_1$, where $\varepsilon_1 = (1-p-\alpha)/2 > 0$, for sufficiently large j. For $i = c$, we obtain $\lim_{j\to\infty}\gamma_{(c,j)} = p - cq$. At this point, condition (4) allows us to conclude that $\gamma_{(c,j)} < -\varepsilon_2$, where $\varepsilon_2 = (cq-p)/2 > 0$, for sufficiently large j. Taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that $\gamma_{(i,j)} < -\varepsilon$, for all except finitely many states. Obviously, $\gamma_{(i,j)}$ are all finite, so Foster's criterion assures that $\{(C_t, O_t); t \geqslant 0\}$ is positive recurrent.

Condition (4) is also necessary for positive recurrence. Indeed, comparing the *Geo*/*Geo*/*c* retrial queue and the standard *Geo*/*Geo*/*c* queue with waiting line, we have that the busy period length of the retrial queue dominates the busy period length of the standard queue in the strong stochastic order sense. However, $p \geqslant cq$ is known to imply instability for the standard *Geo*/*Geo*/*c* queue, so also for the *Geo*/*Geo*/*c* retrial queue. □

We will suppose from now on that condition (4) holds. Then $\{(C_t, O_t); t \geqslant 0\}$ has a unique stationary distribution $(\pi_{ij} : 0 \leqslant i \leqslant c, j \geqslant 0)$. Because the transition matrix $\mathbf{P}$ is spatially non-homogeneous, it seems impossible to compute exactly the probabilities π_{ij}. Thus, in what follows, we study two methods of numerical approximate analysis: direct truncation and generalized truncation (see e.g. Neuts and Rao, 1990). Both methods have been proved effective for dealing with continuous-time retrial queues (see e.g. Falin and Templeton, 1997, Artalejo and Pozo, 2002; Gómez-Corral, 2006).

The method of direct truncation provides a first simple possibility. Roughly speaking, the method consists in placing a fictitious limit K on the orbit. For convenience, we assume $K \geqslant c+1$. The corresponding transition matrix $\overline{\mathbf{P}}(K)$ is identical to the submatrix of $\mathbf{P}$ describing the transitions among levels $l(j)$, for $0 \leqslant j \leqslant K$, except the block $\mathbf{A}_{KK}$ which is replaced by $\mathbf{A}_{KK} + \mathbf{A}_{K,K+1}$. This ensures that $\overline{\mathbf{P}}(K)$ is also stochastic. Thus, we have

$$\overline{\mathbf{P}}(K) = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & & & & & \\ \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A}_{12} & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \mathbf{A}_{c0} & \mathbf{A}_{c1} & \mathbf{A}_{c2} & \cdots & \mathbf{A}_{c,c+1} & & \\ & \mathbf{A}_{c+1,1} & \mathbf{A}_{c+1,2} & \cdots & \mathbf{A}_{c+1,c+1} & \mathbf{A}_{c+1,c+2} & \\ & \ddots & & \ddots & & \ddots & \\ & & \mathbf{A}_{K-1,K-1-c} & \mathbf{A}_{K-1,K-c} & \cdots & \mathbf{A}_{K-1,K-1} & \mathbf{A}_{K-1,K} \\ & & & \mathbf{\Phi}_c(K) & \cdots & \mathbf{\Phi}_1(K) & \mathbf{\Phi}_0(K) \end{pmatrix}, \tag{7}$$

where

$$\mathbf{\Phi}_0(K) = \mathbf{A}_{KK} + \mathbf{A}_{K,K+1}, \tag{8}$$

$$\mathbf{\Phi}_i(K) = \mathbf{A}_{K,K-i}, \quad 1 \leqslant i \leqslant c. \tag{9}$$

Let $(\bar{\pi}_{ij} : 0 \leqslant i \leqslant c, 0 \leqslant j \leqslant K)$ be the stationary distribution of $\overline{\mathbf{P}}(K)$ and $\bar{\pi}(j) = (\bar{\pi}_{0j}, \ldots, \bar{\pi}_{cj})$ be the row vector associated with the level $l(j)$, for $0 \leqslant j \leqslant K$. The vectors $\bar{\pi}(j)$ are determined by solving the system of balance equations

$$(\bar{\pi}(0), \ldots, \bar{\pi}(K))(\mathbf{I} - \overline{\mathbf{P}}(K)) = 0_{(c+1)(K+1)}, \tag{10}$$

and the normalization equation

$$\sum_{k=0}^{K} \bar{\pi}(k) e(c+1) = 1,$$

where $0_{(c+1)(K+1)}$ is the zero row vector of dimension $(c+1)(K+1)$ and $e(c+1)$ is the column vector of 1's with dimension $c+1$.

Set

$$z_0(K) = (\bar{\pi}(0), \ldots, \bar{\pi}(K-1)),$$
$$z_1(K) = \bar{\pi}(K).$$

By partitioning $\mathbf{I} - \overline{\mathbf{P}}(K)$ according to $z_0(K)$ and $z_1(K)$, we have that the system (10) assumes the form

$$(z_0(K), z_1(K)) \begin{pmatrix} \mathbf{B}_{00}(K) & \mathbf{B}_{01}(K) \\ \mathbf{B}_{10}(K) & \mathbf{B}_{11}(K) \end{pmatrix} = (0_{K(c+1)}, 0_{c+1}), \tag{11}$$

where $\mathbf{B}_{00}(K)$, $\mathbf{B}_{01}(K)$, $\mathbf{B}_{10}(K)$ and $\mathbf{B}_{11}(K)$ are block structured matrices with $K \times K$, $K \times 1$, $1 \times K$ and 1×1 blocks of dimension $c+1$, respectively. From (11), we find that

$$z_1(K)\mathbf{B}_{10}(K)\mathbf{B}_{00}^{-1}(K) = -z_0(K), \tag{12}$$

$$z_1(K)(\mathbf{B}_{11}(K) - \mathbf{B}_{10}(K)\mathbf{B}_{00}^{-1}(K)\mathbf{B}_{01}(K)) = 0_{c+1}. \tag{13}$$

It follows further that

$$\mathbf{B}_{00}(K) = \begin{pmatrix} \mathbf{B}_{00}(K-1) & \mathbf{B}_{01}(K-1) \\ \mathbf{C}_0(K-1) & \mathbf{C}_1(K-1) \end{pmatrix},$$

where

$$\mathbf{C}_0(K-1) = (\mathbf{0}_{(c+1)\times(c+1)(K-1-c)}, -\mathbf{A}_{K-1,K-1-c}, \ldots, -\mathbf{A}_{K-1,K-2}),$$
$$\mathbf{C}_1(K-1) = \mathbf{I} - \mathbf{A}_{K-1,K-1}.$$

Until now we have assumed that the orbit level K is arbitrary and fixed. However, the numerical implementation of the approximation based on direct truncation implies the determination of an appropriate orbit level, we say K_f (see Section 6). The successive increment of K (from any initial value to K_f) shows the necessity of computing $\mathbf{B}_{00}^{-1}(K)$ by using blocks associated with the previous orbit level $K-1$. It should be pointed out that $\mathbf{C}_0(K-1) = \mathbf{B}_{10}(K-1)$, so we store up the matrices $\mathbf{B}_{00}(K-1)$, $\mathbf{B}_{01}(K-1)$ and $\mathbf{B}_{10}(K-1)$ and observe that $\mathbf{B}_{00}^{-1}(K)$ has the form (see e.g. Hunter, 1983)

$$\mathbf{B}_{00}^{-1}(K) = \begin{pmatrix} \mathbf{D}_{00} & \mathbf{D}_{01} \\ \mathbf{D}_{10} & \mathbf{D}_{11} \end{pmatrix}, \tag{14}$$

where

$$\mathbf{D}_{00} = (\mathbf{B}_{00}(K-1) - \mathbf{B}_{01}(K-1)\mathbf{C}_1^{-1}(K-1)\mathbf{C}_0(K-1))^{-1},$$

$$\mathbf{D}_{10} = -\mathbf{C}_1^{-1}(K-1)\mathbf{C}_0(K-1)\mathbf{D}_{00},$$

$$\mathbf{D}_{11} = (\mathbf{C}_1(K-1) - \mathbf{C}_0(K-1)\mathbf{B}_{00}^{-1}(K-1)\mathbf{B}_{01}(K-1))^{-1},$$

$$\mathbf{D}_{01} = -\mathbf{B}_{00}^{-1}(K-1)\mathbf{B}_{01}(K-1)\mathbf{D}_{11}.$$

We now may exploit the matrix structure and notice that the sparse block forms of $\mathbf{B}_{01}(K-1)$ and $\mathbf{C}_0(K-1)$ simplify the calculations. In particular, note that the matrix $\mathbf{B}_{01}(K-1)$ has only one non-zero element $-p\bar{q}^c$ in its bottom-right entry. This implies that only the last column of $\mathbf{B}_{00}^{-1}(K-1)\mathbf{B}_{01}(K-1)$ is non-zero and it is obtained by multiplying the last column of $\mathbf{B}_{00}^{-1}(K-1)$ by $-p\bar{q}^c$. Therefore, the key step is the computation of $\mathbf{D}_{00}$. The remaining operations only involve multiplications of known simple matrices and the inversion of matrices of order $c+1$.

The inverse in the definition of $\mathbf{D}_{00}$ can be computed by using small-rank adjustment (see e.g. Hunter (1983)), i.e., if we have the inverse of a matrix $\mathbf{A}$ and we want the inverse of its adjustment $\mathbf{B} = \mathbf{A} + \mathbf{XWY}$, where $\mathbf{W}$ is a matrix of smaller order than $\mathbf{A}$, then we have

$$\mathbf{B}^{-1} = (\mathbf{I} - \mathbf{A}^{-1}\mathbf{X}(\mathbf{W}^{-1} + \mathbf{YA}^{-1}\mathbf{X})^{-1}\mathbf{Y})\mathbf{A}^{-1}.$$

The above formula shows that only the computation of inverses of smaller matrices $\mathbf{W}$ and $\mathbf{W}^{-1} + \mathbf{YA}^{-1}\mathbf{X}$ is needed. In our case, we have $\mathbf{A} = \mathbf{B}_{00}(K-1)$, $\mathbf{X} = -\mathbf{B}_{01}(K-1)$, $\mathbf{W} = \mathbf{C}_1^{-1}(K-1)$ and $\mathbf{Y} = \mathbf{C}_0(K-1)$. Thus, we obtain that $\mathbf{D}_{00} = \mathbf{B}^{-1} = (\mathbf{I} - \mathbf{D}_{01}\mathbf{C}_0(K-1))\mathbf{B}_{00}^{-1}(K-1)$, so $\mathbf{D}_{00}$ is obtained by multiplications and additions of already computed matrices.

Now the computation of vector $z_1(K)$ is reduced to solve the system (13) subject to the normalizing condition

$$\bar{\pi}(K)(e(c+1) - \mathbf{B}_{10}(K)\mathbf{B}_{00}^{-1}(K)e(K(c+1))) = 1, \tag{15}$$

i.e., the resulting set of equations to solve consists only of $c+1$ linearly independent equations. Finally, $z_0(K)$ follows from (12). Summarizing, we have the following algorithm.

Algorithm 3 (*Approximate computation of the stationary distribution using truncation*)

$K := c+1$
Compute $\mathbf{B}_{00}^{-1}(K)$
Compute $z_1(K)$ by (13) and (15), and $z_0(K)$ by (12)
Store $\mathbf{B}_{00}(K)$, $\mathbf{B}_{00}^{-1}(K)$, $\mathbf{B}_{01}(K)$ and $\mathbf{B}_{10}(K)$
$K := K+1$
While $K \leqslant K_f$ do
  Compute $\mathbf{B}_{00}^{-1}(K)$ by (14)
  Compute $z_1(K)$ by (13) and (15), and $z_0(K)$ by (12)
  Update $\mathbf{B}_{00}(K)$, $\mathbf{B}_{00}^{-1}(K)$, $\mathbf{B}_{01}(K)$ and $\mathbf{B}_{10}(K)$
  $K := K+1$
od

A second approximation for the stationary distribution of the system state can be obtained by using generalized truncation. According to this approximation, from a level $l(K)$ and up, only K customers in orbit are permitted to conduct retrials. This implies that the new transition matrix $\widetilde{\mathbf{P}}(K)$ has a homogeneous $GI/M/1$

structure with a large number of boundary states corresponding to levels $l(j)$, for $0 \leqslant j \leqslant K-1$. Let $(\tilde{\pi}_{ij} : 0 \leqslant i \leqslant c, j \geqslant 0)$ be the stationary distribution of $\widetilde{\mathbf{P}}(K)$ and $\tilde{\pi}(j) = (\tilde{\pi}_{0j}, \ldots, \tilde{\pi}_{cj})$ be the row vector with the stationary probabilities of the jth level. It is well-known that the stationary distribution is matrix geometric, i.e., we have

$$\tilde{\pi}(j) = \tilde{\pi}(K)\mathbf{R}^{j-K}, \quad j \geqslant K+1, \tag{16}$$

where $\mathbf{R}$ is the minimal non-negative solution of the matrix equation

$$\mathbf{R} = \sum_{k=0}^{c+1} \mathbf{R}^k \mathbf{A}_{K,K+1-k}.$$

The matrix $\mathbf{R}$ can be computed numerically by using general-purpose iterative schemes, based only on the general form of $\widetilde{\mathbf{P}}(K)$ and without any further assumption, except that the underlying Markov chain is irreducible and positive recurrent (see e.g. Neuts, 1981). To be concrete, by starting somewhat arbitrarily with $\mathbf{R}(0) = \mathbf{0}_{(c+1)\times(c+1)}$, we may iteratively evaluate the sequence $\{\mathbf{R}(n); n \geqslant 1\}$ as follows:

$$\mathbf{R}(n+1) = \left(\mathbf{A}_{K,K+1} + \sum_{k=2}^{c+1} \mathbf{R}^k(n)\mathbf{A}_{K,K+1-k}\right)(\mathbf{I} - \mathbf{A}_{KK})^{-1}, \quad n \geqslant 0.$$

To define a stopping criterion, the standard practice is to keep track of two successive matrices in each iteration and stop iterations when $\|\mathbf{R}(n) - \mathbf{R}(n-1)\|_\infty < \varepsilon$.

Instead of the above classical iterative method, we next take advantage of the special matrix structure for $\mathbf{A}_{K,K+1}$, which has the form

$$\mathbf{A}_{K,K+1} = p\bar{q}^c e'_{c+1}(c+1)e_{c+1}(c+1),$$

where $e_i(j)$ is a row vector of dimension j such that all its entries are equal to 0, except for the ith one which is equal to 1. Straightforward algebra, which mostly repeats arguments of Theorem 8.5.2 of Latouche and Ramaswami (1999), allows us to prove that $\mathbf{R}$ can be explicitly determined, once its spectral radius $\eta(K)$ is known, as

$$\mathbf{R} = p\bar{q}^c \begin{pmatrix} \mathbf{0}_{c\times(c+1)} \\ u \end{pmatrix},$$

where u is the left eigenvector of $\mathbf{R}$ corresponding to $\eta(K)$. Indeed, we notice that $\eta(K) = p\bar{q}^c(u)_{c+1}$, where $(u)_{c+1}$ is the $(c+1)$th entry of u. The matrix $\mathbf{R}$ can be also expressed as

$$\mathbf{R} = \mathbf{A}_{K,K+1}\left(\mathbf{I} - \sum_{k=1}^{c+1} \eta^{k-1}(K)\mathbf{A}_{K,K+1-k}\right)^{-1}.$$

Thus, the problem reduces to determine the spectral radius $\eta(K)$ and its eigenvector u. One appropriate procedure for computing $\eta(K)$ is to solve the equation

$$\det\left[\eta(K)\mathbf{I} - \sum_{k=0}^{c+1} \eta^k(K)\mathbf{A}_{K,K+1-k}\right] = 0,$$

which leads to the computation of $\eta(K)$ as a root in $(0,1)$ of a polynomial equation by applying an elementary procedure such as the bisection or the secant method. Once $\eta(K)$ is numerically computed, u is determined as the solution to

$$u\left(\eta(K)\mathbf{I} - \sum_{k=0}^{c+1} \eta^k(K)\mathbf{A}_{K,K+1-k}\right) = 0_{c+1},$$

which satisfies $\eta(K) = p\bar{q}^c(u)_{c+1}$.

It remains to compute the vectors $\tilde{\pi}(j)$, for $0 \leqslant j \leqslant K$. The censored Markov chain in the set $l(0) \cup \cdots \cup l(K)$ has a transition matrix of the form (7) with $\mathbf{\Phi}_i(K)$ given by

$$\mathbf{\Phi}_i(K) = \sum_{k=0}^{c-i} \mathbf{R}^k \mathbf{A}_{K,K-i-k}, \quad 0 \leqslant i \leqslant c. \tag{17}$$

Therefore, we can follow the ideas that we described above. The algorithm is still valid with four modifications. First, we have to compute $\mathbf{\Phi}_i(K)$, for $0 \leqslant i \leqslant c$, by (17) instead of (8), (9). Second, we notice that $\mathbf{C}_0(K-1) \neq \mathbf{B}_{10}(K-1)$. As a result, we need to compute $\mathbf{C}_0(K-1)$ instead of keeping $\mathbf{B}_{10}(K-1)$. Third, we must consider the normalization condition given by

$$\tilde{\pi}(K)((\mathbf{I}-\mathbf{R})^{-1}e(c+1) - \mathbf{B}_{10}(K)\mathbf{B}_{00}^{-1}(K)e(K(c+1))) = 1.$$

Finally, we have to compute $\tilde{\pi}(j)$, for $j \geqslant K+1$, by (16), up to any desired level of accuracy.

It should be noted that the positive recurrence of $\widetilde{\mathbf{P}}(K)$ amounts to $\eta(K) < 1$. Thus, in order to select K_f, we should start with an initial value $K \geqslant c+1$ such that $\eta(K) < 1$.

4. Busy period analysis

In this section, we present some algorithms for the recursive computation of the busy period characteristics. Let $L_{(i,j)}$ be a random variable representing the first passage time to $(0,0)$, starting from a state (i,j). Then, the busy period is defined as $L = L_{(1,0)}$. We are interested in obtaining the probabilities $P\{L_{(i,j)} = k\}$, for $k \geqslant 1$, and the moments of L. Conditioning on the first transition of the process, we have that

$$P\{L_{(i,j)} = 1\} = P_{(i,j)(0,0)}, \quad 0 \leqslant i \leqslant c,\ j \geqslant 0, (i,j) \neq (0,0), \tag{18}$$

$$P\{L_{(i,j)} = k\} = \sum_{(m,n)\in S_{(i,j)}-\{(0,0)\}} P_{(i,j)(m,n)} P\{L_{(m,n)} = k-1\}, \quad 0 \leqslant i \leqslant c,\ j \geqslant 0,\ (i,j) \neq (0,0),\ k \geqslant 2. \tag{19}$$

For every $k \geqslant 1$, we define the infinite column vector $l^k = (l^k(0), l^k(1), \ldots)'$ where

$$l^k(0) = (P\{L_{(1,0)} = k\}, \ldots, P\{L_{(c,0)} = k\})',$$
$$l^k(j) = (P\{L_{(0,j)} = k\}, \ldots, P\{L_{(c,j)} = k\})', \quad j \geqslant 1.$$

We denote by $\mathbf{P}_L$ the matrix that results by removing the first row and the first column of the transition matrix $\mathbf{P}$. Let also P_0 be the column vector of dimension $c + (c+1)K$ created by removing the first element of the first column of the matrix $\mathbf{P}$. Then Eqs. (18) and (19) are written in matrix form as

$$\begin{aligned} l^1 &= P_0, \\ l^k &= \mathbf{P}_L l^{k-1}, \quad k \geqslant 2. \end{aligned} \tag{20}$$

The probability $P\{L = k\}$ equals the first element of $l^k(0)$, i.e., $P\{L = k\} = e_1(c) l^k(0)$.

For any fixed k, the computation of $l^k(0)$ can be done recursively by exploiting (20). Indeed, because of the lower Hessenberg matrix structure of $\mathbf{P}_L$, (20) assumes the block form:

$$l^k(0) = \mathbf{A}_{00}^* l^{k-1}(0) + \mathbf{A}_{01}^* l^{k-1}(1), \tag{21}$$

$$l^k(j) = \mathbf{A}_{j0}^* l^{k-1}(0) + \sum_{i=1}^{j+1} \mathbf{A}_{ji} l^{k-1}(i), \quad 1 \leqslant j \leqslant c, \tag{22}$$

$$l^k(j) = \sum_{i=j-c}^{j+1} \mathbf{A}_{ji} l^{k-1}(i), \quad j \geqslant c+1, \tag{23}$$

where $\mathbf{A}_{00}^*$ is obtained from $\mathbf{A}_{00}$ by deleting the first row and column. Analogously, $\mathbf{A}_{01}^*$ follows from $\mathbf{A}_{01}$ by deleting the first row, while the matrices $\mathbf{A}_{j0}^*$, for $1 \leqslant j \leqslant c$, are obtained from $\mathbf{A}_{j0}$ by removing the first column.

Observe that to compute $l^k(0)$ we need $l^{k-1}(0)$ and $l^{k-1}(1)$ so we need $l^{k-2}(0)$, $l^{k-2}(1)$ and $l^{k-2}(2)$, etc. Hence, for the computation of $P\{L = k\}$ we need to begin by computing the vectors $l^1(0), l^1(1), \ldots, l^1(k-1)$. It should be pointed out that it is impossible to move in one transition from states with at least one customer in orbit to the state (0,0). Hence $l^1(j) = 0$, for $j \geqslant 1$, and using (23) we find that $l^i(j) = 0$, for $i \geqslant 1$ and $j \geqslant (i-1)c + 1$.

We next organize the above facts to formulate the following algorithm for the computation of $P\{L = k\}$.

Algorithm 4 (*Exact computation of* $P\{L = k\}$, *for any given* $k \geqslant 1$)

```
l^1(0) := P_0(0)
For j := 1 to k − 1 do
    l^1(j) := 0
od
For i := 2 to k do
    For j := 0 to min{k − i, (i − 1)c} do
        Compute l^i(j) using (21), (23)
    od
    For j := min{k − i, (i − 1)c} + 1 to k − i do
        l^i(j) := 0
    od
od
P{L = k} := e_1(c) l^k(0)
```

For a given level of accuracy $\varepsilon > 0$, we may compute $P\{L = k\}$ until the $(1-\varepsilon)$-percentile of L has been achieved, i.e., we stop the computations at a level K such that $P\{L \leqslant K-1\} \leqslant 1-\varepsilon < P\{L \leqslant K\}$. The idea now is to approximate the moment $E[L^n]$ by the truncated moment $\sum_{k=0}^{K} k^n P\{L = k\}$. However, this method neglects small probabilities $P\{L = k\}$ which are multiplied by large numbers k^n. An alternative approach may be attained by considering the direct truncation model with transition matrix $\overline{\mathbf{P}}(K)$. If we denote by $L^{(K)}$ the length of a busy period in the truncated model, then we have

$$P\{L = k\} = P\{L^{(K)} = k\}, \quad 0 \leqslant k \leqslant K.$$

This happens because the event $\{L = k\}$ with $k \leqslant K$ implies that the process $\{(C_t, O_t);\ t \geqslant 0\}$ has not visited $l(K)$ during the busy period under consideration. Then, the transition probabilities of the paths that are counted in $\{L = k\}$ are equal either if we use the matrix $\mathbf{P}$ or its truncation $\overline{\mathbf{P}}(K)$. It seems numerically advantageous to approximate the moments of L by the moments of $L^{(K)}$. Note that we can think of $L^{(K)}$ as a discrete PH distribution with representation $PH_d(e_1(c + (c+1)K), \overline{\mathbf{P}}_L(K))$, where $\overline{\mathbf{P}}_L(K)$ is obtained by neglecting the first row and column of $\overline{\mathbf{P}}(K)$. Using standard results for the moments of a discrete PH distribution (see e.g. Latouche and Ramaswami, 1999), we have that the factorial moments of $L^{(K)}$ are given by

$$E[L^{(K)}(L^{(K)}-1)\ldots(L^{(K)}-k+1)] = k!e_1(c+(c+1)K)(\mathbf{I} - \overline{\mathbf{P}}_L(K))^{-k}\overline{\mathbf{P}}_L(K)^{k-1}e(c+(c+1)K),$$
$$k \geqslant 1. \tag{24}$$

For using (24) we need to compute the column vectors $(\mathbf{I} - \overline{\mathbf{P}}_L(K))^{-k}\overline{\mathbf{P}}_L(K)^{k-1}\ e(c+(c+1)K)$. It amounts to successively solve linear systems of the form $(\mathbf{I} - \overline{\mathbf{P}}_L(K))x = a$. This can be done at low computational cost by exploiting the lower block Hessenberg form of the coefficient matrix $\mathbf{I} - \overline{\mathbf{P}}_L(K)$. In particular, we can use a simplified block Gaussian elimination procedure (for more details see e.g. Stewart, 1994).

5. Waiting time analysis

Let W be a random variable representing the waiting time that a tagged customer spends in orbit. According to this definition, W excludes the service time. We remark (see Section 2) that a primary arrival has priority over customers in the retrial orbit and assume the RO policy among retrying customers. In what follows, we approximate the analysis of W in the intractable *Geo*/*Geo*/*c* retrial queue by the parallel study for the truncated model with sufficiently large K.

Since we deal with the G-EAS policy, the probability that a primary arrival customer finds the state (c,j) is $\bar{q}^c\bar{\pi}_{cj}$. Let $W_{(i,j)}$ be the conditional waiting time for a customer in orbit, given that the current system state is (i,j). We now may condition on the state viewed by the tagged customer. This yields

$$P\{W=k\} = \bar{q}^c \sum_{j=0}^{K-1} \bar{\pi}_{cj} P\{W_{(c,j+1)} = k\}, \quad k \geqslant 1.$$

Moreover, we have

$$P\{W=0\} = 1 - \bar{q}^c \sum_{j=0}^{K-1} \bar{\pi}_{cj}.$$

The probabilities $P\{W_{(i,j)} = k\}$ can be obtained recursively by conditioning on the state of the next transition. We have the first-step analysis equations

$$P\{W_{(i,j)} = 1\} = \sum_{(m,n)\in S^*_{(i,j)}} P_{(i,j)(m,n)} \frac{j-n}{j}, \quad 0 \leqslant i \leqslant c, 1 \leqslant j \leqslant K, \tag{25}$$

$$\begin{aligned} P\{W_{(i,j)} = k\} = &\sum_{(m,n)\in S^*_{(i,j)}} P_{(i,j)(m,n)} \frac{n}{j} P\{W_{(m,n)} = k-1\} + \delta_{ic}(1-\delta_{jK}) P_{(c,j)(c,j+1)} P\{W_{(c,j+1)} = k-1\} \\ &+ \delta_{ic}\delta_{jK} P_{(c,K)(c,K+1)} P\{W_{(c,K)} = k-1\}, \quad 0 \leqslant i \leqslant c, 1 \leqslant j \leqslant K, k \geqslant 2, \end{aligned} \tag{26}$$

where $S^*_{(i,j)} = S_{(i,j)}$, for $0 \leqslant i \leqslant c-1$, and $S^*_{(c,j)} = S_{(c,j)} - \{(c,j+1)\}$.

The above Eqs. (25) and (26) provide a simple recursive scheme to obtain $P\{W=k\}$ up to any given level of accuracy. In particular, they can be expressed in matrix form as in the case of (18) and (19) for $P\{L=k\}$. Thus, a similar algorithm can be developed. By multiplying (25) by z and (26) by z^k, and adding for all k, we obtain recursive relations for the corresponding probability generating functions $W_{(i,j)}(z) = \sum_{k=1}^{\infty} P\{W_{(i,j)} = k\} z^k$:

$$\begin{aligned} W_{(i,j)}(z) = &\sum_{(m,n)\in S^*_{(i,j)}} P_{(i,j)(m,n)} \left(\frac{j-n}{j} z + \frac{n}{j} z W_{(m,n)}(z) \right) + \delta_{ic}(1-\delta_{jK}) P_{(c,j)(c,j+1)} z W_{(c,j+1)}(z) \\ &+ \delta_{ic}\delta_{jK} P_{(c,K)(c,K+1)} z W_{(c,K)}(z), \quad 0 \leqslant i \leqslant c,\ 1 \leqslant j \leqslant K. \end{aligned}$$

By differentiating at $z=1$, we get recursive relations for the factorial conditional waiting time moments $W^{(k)}_{(i,j)} = E[W_{(i,j)} \ldots (W_{(i,j)} - k + 1)]$, for $k \geqslant 1$. In particular, we have

$$\begin{aligned} W^{(k)}_{(i,j)} = &\sum_{(m,n)\in S^*_{(i,j)}} P_{(i,j)(m,n)} \left(\frac{j-n}{j} \delta_{k1} + \frac{n}{j} \left(W^{(k)}_{(m,n)} + k W^{(k-1)}_{(m,n)} \right) \right) + \delta_{ic}(1-\delta_{jK}) P_{(c,j)(c,j+1)} \left(W^{(k)}_{(c,j+1)} + k W^{(k-1)}_{(c,j+1)} \right) \\ &+ \delta_{ic}\delta_{jK} P_{(c,K)(c,K+1)} \left(W^{(k)}_{(c,K)} + k W^{(k-1)}_{(c,K)} \right), \quad 0 \leqslant i \leqslant c,\ 1 \leqslant j \leqslant K,\ k \geqslant 1, \end{aligned}$$

and consequently

$$E[W] = \bar{q}^c \sum_{j=0}^{K-1} \bar{\pi}_{cj} W^{(1)}_{(c,j+1)}.$$

6. Numerical results

Next we present numerical examples to illustrate the behavior of the $Geo/Geo/c$ retrial queue under study. First, our interest is in the selection criterion of the level K_f. It is clear that a suitably chosen level K_f for a particular performance measure could be inappropriate for another measure. Thus, we suggest to focus on a criterion based on the spectral radius $\eta(K)$. We propose to start with an initial value $K \geqslant c+1$ and progressively increase the value of K until the change in $\eta(K)$ due to an increase in K is sufficiently small. To be more precise, we choose the smallest value K_f with relative error

$$\left|1-\frac{\eta(K_f-1)}{\eta(K_f)}\right|<\varepsilon, \tag{27}$$

where $\varepsilon > 0$ is an arbitrary small value.

With this selection criterion, Tables 1 and 2 list the appropriate values of K_f, the blocking probability B and the mean number of customers in orbit $E[O]$, for queues with $c = 5$, $q = 0.2$, different values of p and s, and

Table 1
The blocking probability

s		$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$
0.1	K_f	11	18	23	29
	$R_T(K_f)$	1.112×10^{-11}	5.652×10^{-12}	1.348×10^{-9}	2.062×10^{-6}
	$R_{NR}(K_f)$	7.563×10^{-13}	1.071×10^{-13}	1.269×10^{-11}	1.171×10^{-8}
	B	0.00151	0.03010	0.14869	0.43449
0.3	K_f	7	9	11	13
	$R_T(K_f)$	9.847×10^{-8}	1.555×10^{-6}	0.00002	0.00069
	$R_{NR}(K_f)$	5.830×10^{-9}	3.670×10^{-8}	2.831×10^{-7}	4.862×10^{-6}
	B	0.00154	0.03134	0.15562	0.44921
0.5	K_f	7	7	7	8
	$R_T(K_f)$	1.025×10^{-7}	0.00002	0.00081	0.00545
	$R_{NR}(K_f)$	2.133×10^{-9}	4.186×10^{-7}	0.00001	0.00005
	B	0.00156	0.03212	0.15986	0.45804
0.7	K_f	7	7	7	7
	$R_T(K_f)$	1.016×10^{-7}	0.00002	0.00075	0.00794
	$R_{NR}(K_f)$	4.540×10^{-10}	8.641×10^{-8}	2.232×10^{-6}	0.00002
	B	0.00158	0.03266	0.16278	0.46391
0.9	K_f	7	7	7	7
	$R_T(K_f)$	1.003×10^{-7}	0.00002	0.00072	0.00755
	$R_{NR}(K_f)$	1.669×10^{-11}	3.111×10^{-9}	7.719×10^{-8}	8.240×10^{-7}
	B	0.00159	0.03307	0.16492	0.46804

Table 2
The mean number of customers in orbit

s		$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$
0.1	K_f	11	18	23	29
	$R_T(K_f)$	3.235×10^{-10}	8.510×10^{-11}	1.466×10^{-8}	0.00001
	$R_{NR}(K_f)$	1.907×10^{-11}	1.656×10^{-12}	1.944×10^{-10}	3.055×10^{-7}
	$E[O]$	0.00108	0.04660	0.41150	2.41272
0.3	K_f	7	9	11	13
	$R_T(K_f)$	3.093×10^{-6}	0.00002	0.00028	0.00594
	$R_{NR}(K_f)$	1.130×10^{-7}	4.928×10^{-7}	4.040×10^{-6}	0.00010
	$E[O]$	0.00042	0.01997	0.19030	1.19400
0.5	K_f	7	7	7	8
	$R_T(K_f)$	4.163×10^{-6}	0.00043	0.00782	0.03843
	$R_{NR}(K_f)$	4.674×10^{-8}	5.500×10^{-6}	0.00013	0.00081
	$E[O]$	0.00029	0.01448	0.14506	0.95982
0.7	K_f	7	7	7	7
	$R_T(K_f)$	5.083×10^{-6}	0.00050	0.00847	0.05742
	$R_{NR}(K_f)$	1.048×10^{-8}	1.141×10^{-6}	0.00002	0.00034
	$E[O]$	0.00023	0.01209	0.12574	0.86713
0.9	K_f	7	7	7	7
	$R_T(K_f)$	5.850×10^{-6}	0.00055	0.00898	0.05909
	$R_{NR}(K_f)$	3.665×10^{-10}	3.614×10^{-8}	7.229×10^{-7}	8.598×10^{-6}
	$E[O]$	0.00020	0.01077	0.11543	0.82221

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$\varepsilon = 10^{-3}$. In both tables, $R_T(K_f)$ and $R_{NR}(K_f)$ denote the relative errors associated with a direct truncation and the generalized truncation suggested by Neuts and Rao (1990), respectively. Thus, similarly to (27), $R_T(K_f)$ and $R_{NR}(K_f)$ are evaluated in Table 1 (respectively, Table 2) accordingly to those approximate values for B (respectively, $E[O]$) obtained from $\overline{\mathbf{P}}(K)$ and $\widetilde{\mathbf{P}}(K)$ by setting $K = K_f - 1$ and K_f. In all cases, we observe that $R_{NR}(K_f) < R_T(K_f)$, so that we conclude that a generalized truncation seems to provide a better approximation for the stationary characteristics of the $Geo/Geo/c$ retrial queue. Hence, the values for B and $E[O]$ listed in the tables are those obtained by applying a generalized truncation at the level K_f. Note that, for fixed values of s, B and $E[O]$ are increasing functions of p. For fixed values of p, B increases with increasing values of s and, in contrast, $E[O]$ is a decreasing function of s.

In Table 3, we present the exact mass function of the length L of a busy period for various choices of the pair (p, s). At the light of our numerical experience including examples that are not reported here, we point out that $P\{L = k\}$ seems to be a decreasing function of k, irrespective of the magnitudes of p and s. Nevertheless, Table 3 shows that the rate of convergence of $P\{L = k\}$ towards zero does notably depend on the value of p. In particular, for fixed values of s, the slower convergence is associated with the choice of higher values of p. In Table 4, we approximate the moments $E[L]$ and $Var(L)$ by using the values $E[L^{(K_f)}]$ and $Var(L^{(K_f)})$, respectively, with those levels K_f listed in Table 1. As was expected, both descriptors increase with increasing values of p, when we fix s. In contrast, they decrease with increasing values of s, once we fix p, with differences of magnitude more apparent in the case of higher values of p.

Table 3
The mass function of the busy period

k	$(p,s) = (0.2, 0.1)$	$(p,s) = (0.2, 0.9)$	$(p,s) = (0.8, 0.1)$	$(p,s) = (0.8, 0.9)$
1	0.16000	0.16000	0.03999	0.03999
2	0.11392	0.11392	0.01792	0.01792
3	0.08730	0.08730	0.01068	0.01068
4	0.07038	0.07038	0.00741	0.00741
5	0.05877	0.05877	0.00566	0.00566
6	0.05029	0.05029	0.00462	0.00462
7	0.04380	0.04380	0.00394	0.00395
8	0.03863	0.03863	0.00348	0.00349
9	0.03439	0.03439	0.00315	0.00317
10	0.03081	0.03081	0.00290	0.00293
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
146	1.602×10^{-7}	1.370×10^{-7}	6.115×10^{-4}	1.353×10^{-3}
147	1.467×10^{-7}	1.252×10^{-7}	6.106×10^{-4}	1.351×10^{-3}
148	1.343×10^{-7}	1.144×10^{-7}	6.096×10^{-4}	1.348×10^{-3}
149	1.230×10^{-7}	1.045×10^{-7}	6.087×10^{-4}	1.345×10^{-3}
150	1.126×10^{-7}	9.552×10^{-8}	6.079×10^{-4}	1.342×10^{-3}
$P\{L \leqslant 150\}$	0.99999	0.99999	0.22767	0.32783

Table 4
Mean and variance of $L^{(K_f)}$

s		$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$
0.1	$E[L^{(K_f)}]$	9.46730	22.64749	86.88340	1135.99651
	$Var(L^{(K_f)})$	109.54890	754.33596	11530.15448	1682179.59557
0.3	$E[L^{(K_f)}]$	9.46014	22.15289	74.22936	560.08671
	$Var(L^{(K_f)})$	108.99381	692.91946	7659.65627	383643.55288
0.5	$E[L^{(K_f)}]$	9.45913	22.07519	72.19682	487.87305
	$Var(L^{(K_f)})$	108.93164	684.78211	7148.27291	286688.58022
0.7	$E[L^{(K_f)}]$	9.45878	22.04737	71.44458	461.90924
	$Var(L^{(K_f)})$	108.91167	681.97266	6967.03397	255490.27774
0.9	$E[L^{(K_f)}]$	9.45862	22.03391	71.07639	450.61853
	$Var(L^{(K_f)})$	108.90239	680.63252	6879.93031	242618.81403

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Table 5
The mass function of the waiting time

k	$(p,s)=(0.2,0.1)$	$(p,s)=(0.2,0.9)$	$(p,s)=(0.8,0.1)$	$(p,s)=(0.8,0.9)$
0	0.99950	0.99947	0.85762	0.85134
1	2.920×10^{-5}	2.687×10^{-4}	0.00436	0.03169
2	3.682×10^{-5}	1.353×10^{-4}	0.00559	0.02347
3	3.770×10^{-5}	6.168×10^{-5}	0.00595	0.01723
4	3.617×10^{-5}	2.869×10^{-5}	0.00597	0.01308
5	3.375×10^{-5}	1.382×10^{-5}	0.00584	0.01020
6	3.107×10^{-5}	6.880×10^{-6}	0.00564	0.00813
7	2.837×10^{-5}	3.515×10^{-6}	0.00540	0.00659
8	2.579×10^{-5}	1.836×10^{-6}	0.00515	0.00541
9	2.338×10^{-5}	9.777×10^{-7}	0.00491	0.00450
10	2.115×10^{-5}	5.288×10^{-7}	0.00466	0.00377
11	1.911×10^{-5}	2.899×10^{-7}	0.00443	0.00319
12	1.725×10^{-5}	1.609×10^{-7}	0.00420	0.00271
13	1.556×10^{-5}	9.025×10^{-8}	0.00399	0.00232
14	1.403×10^{-5}	5.109×10^{-8}	0.00379	0.00200
15	1.264×10^{-5}	2.917×10^{-8}	0.00359	0.00173
$P\{W\leqslant 15\}$	0.99988	0.99999	0.93117	0.98743

Table 6
Mean and variance of W

s		$p=0.2$	$p=0.4$	$p=0.6$	$p=0.8$
0.1	$E[W]$	0.00541	0.11650	0.68584	3.01579
	$Var(W)$	0.10517	2.39483	16.72482	115.10389
0.3	$E[W]$	0.00212	0.04994	0.31710	1.47391
	$Var(W)$	0.01400	0.38694	3.33642	28.33867
0.5	$E[W]$	0.00146	0.03621	0.24019	1.10087
	$Var(W)$	0.00619	0.19862	1.94009	16.13409
0.7	$E[W]$	0.00117	0.03023	0.20810	0.95268
	$Var(W)$	0.00392	0.14044	1.51183	12.47816
0.9	$E[W]$	0.00101	0.02692	0.19095	0.90038
	$Var(W)$	0.00294	0.11469	1.32479	11.55164

Tables 5 and 6 show the influence of p and s on the mass function, the mean and the variance of W. To that end, we approximate these measures through the corresponding characteristics derived by using the truncation levels K_f given in Table 1. Table 5 lists values of $P\{W=k\}$ for four choices of the pair (p,s). The mass function appears to be a decreasing function of k in the cases $(p,s)\in\{(0.2,0.9),(0.8,0.9)\}$ and exhibits a different behavior when dealing with $(p,s)\in\{(0.2,0.1),(0.8,0.1)\}$. In such cases, the mass function corresponds to a two-modal distribution, with the first mode at $k=0$. We observe from Table 6 that, for fixed values of s, the mean and the variance of W increase with increasing values of p. On the contrary, both descriptors are decreasing functions of s, for each fixed value of p. Heavier tails correspond to increase p and decrease s, thus implying that the case $(p,s)=(0.8,0.1)$ with a higher congestion has heavier tails in our numerical examples.

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