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Decision Support

Satisfying optimization method based on goal programming for fuzzy multiple objective optimization problem

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ABSTRACT

This paper proposes a satisfying optimization method based on goal programming for fuzzy multiple objective optimization problem. The aim of this presented approach is to make the more important objective achieving the higher desirable satisfying degree. For different fuzzy relations and fuzzy importance, the reformulated optimization models based on goal programming is proposed. Not only the satisfying results of all the objectives can be acquired, but also the fuzzy importance requirement can be simultaneously actualized. The balance between optimization and relative importance is realized. We demonstrate the efficiency, flexibility and sensitivity of the proposed method by numerical examples.

1. Introduction

In actual decision making situations, a major concern is that most decision problems involve multiple criteria (attributes or objectives). During the recent years, multiple objective decision making (MODM) problem as the crucial part of multiple criteria decision making (MCDM) has become a promising field, and attracted more and more researchers (Steuer, 1986; Yu, 1985). However, in real-world, synchronous optimization of multiple objectives is an iterative, trial-and-error process for being conflicting, incommensurable and imprecise or fuzzy in nature. This often leads to that the ultimate goal in MODM is to seek a most preferred compromise solution rather than optimal one. During the last three decades, different methods have been employed to solve MODM problem (Chanking and Haimes, 1983; Lai and Hwang, 1994).

However, much of decision making in the real-world takes place in an environment where the objectives, constraints or parameters are not known precisely (Liu, 2002). Therefore a decision is often made on the basis of vague information or uncertain data. In 1970, Zadeh and Bellman introduced the fuzzy set theory into the traditional decision making problem involving uncertainty and imprecision. According to the fuzzy theory, the inaccurate objectives and constraints, called fuzzy objectives and constraints, are represented by associated membership functions. Tanaka et al. (1974) initially proposed the concept of fuzzy mathematical programming, and Zimmermann (1978) formulated fuzzy linear programming with several objectives. Moreover, the solution of multiple objective optimization problem is dependent upon the DM's preference. This can be represented by relative importance and priority (Lin, 2004; Tiwari et al., 1986, 1987) besides an explicit utility function or progressive articulation in actual decision making (Sakawa et al., 2004; Sakawa and Yauchi, 2001; Yang and Li, 2002; Yang and Sen, 1996). Transforming all objective functions into a scalar criterion by weights is conventional strategy for importance preference, which is based on p-norm ($1 \leqslant p \leqslant \infty$) formulation (Yang, 2000). The weighted additive model can be referred to in many studies (Chen and Tsai, 2001; Hannan, 1981; Tiwari et al., 1987), which is equivalent to 1-norm. In Tiwari et al. (1987), the DM assigns different weights as coefficients of the individual terms in the simple additive fuzzy achievement function to reflect their relative importance. In addition, there are several weighted min–max approaches on ∞-norm (Lin, 2004; Wang et al., 2001; Yang and Li, 2002).

In reality, however, it is difficult to specify the weights for DM since he or she might has vague or imprecise knowledge about these objectives, constraints and the environment in advance. For assessing the fuzzy importance of the objectives, Narasimhan (1980) has used linguistic terms, such as "very important" and "moderately important". Chen and Tsai (2001) distinguish the relative importance between objectives through determining a desirable achievement degree for each objective. That is, the more important the objective, the higher the desirable achievement degree. They added the inequity about membership function and desirable achievement degree of each objective to the model as new constraint in an explicit way. In order to express the fuzzy importance relations, Aköz and Petrovic (2007) define three types of fuzzy binary relation for the different linguistic terms, such as

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"slightly more important than", "moderately more important" and "significantly more important than".

Whereas, the additional constraints may be too strict for the optimization model to acquire the satisfying even feasible solution when the DM requires a higher desirable achievement degree requirement for a special fuzzy objective in the studies of Chen and Tsai (2001). The determination of distinct desirable satisfying degrees may not be possible when handling the real life problem. Furthermore, the aim of their reformulation is to maximize the sum of the achievement degree. The fuzzy relation '$\cong$' will also result in only one feasible solution for reasons of the existence of the membership constraint $\mu_{f_i}(\boldsymbol{x}) \leqslant 1$. Thus they cannot deal with '$\cong$' but for '$\lesssim$' and '$\gtrsim$'. Although the strict comparison constraints between the membership functions expressing preemptive priorities are replaced by the imprecise goal hierarchy in Aköz and Petrovic (2007), i.e. the relation between the satisfying degrees of the goals and the fuzzy importance, the method is not able to solve the problems with euqality fuzzy relation '$\cong$' either. Simultaneously the proper satisfying degrees of the fuzzy importance must be given ahead, which need more decision procedures. Moreover there may be the inferior solution by means of the goal hierarchy method. Goal Programming (GP) (Charnes and Cooper, 1961; Ijiri, 1965; Narasimhan, 1980; Pal and Moitra, 2003) as the most promising methodologies for MODM, has been utilized in real-world decision making problems. GP, initially introduced by Charnes and Cooper (1961), is used to consider all the objectives with different attainment relations in finding an acceptable solution through minimizing the deviations from the expected values. Accordingly, we introduce a satisfying optimization method based on goal programming in this paper. It is adapted to solve the optimization problems with the above three types of fuzzy relations. Following the more important objective achieving the higher desirable satisfying degrees, fuzzy multiple objective optimization problem is reformulated. In the new model, both of all the desirable achievement degrees and the importance difference between the objectives are maximized by ranking the desirable satisfying degrees under the interaction with DM. The results of all the objectives are not only satisfying to DM, but also consistent with his or her fuzzy preference. The trade-off between optimization and importance requirement is realized.

This paper is organized as follows: In Section 2 fuzzy multiple objective optimization problem is described. Section 3 presents the satisfying optimization method based on goal programming. The optimization algorithm is provided in Section 4. We demonstrate the efficiency, flexibility and sensitivity of the proposed optimization approach and obtain the minimum parameter for limit case by the numerical examples in Section 5. Section 6 makes the conclusions.

2. Fuzzy multiple objective optimization problem

2.1. Multiple objective optimization problem

A multiple objective optimization problem can, in general, be represented as follows:

$$\begin{cases} \text{opt} & (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})), \\ \text{s.t.} & \boldsymbol{x} \in G = \{\boldsymbol{x} | g_j(\boldsymbol{x}) \leqslant 0, j = 1, \ldots, m\}, \end{cases} \tag{1}$$

where "*opt*" denotes minimization or maximization; $\boldsymbol{x} = (\boldsymbol{x}_1, x_2, \ldots, x_n)$ is decision vector; $f_i(\boldsymbol{x})$, $(i = 1, \ldots, k)$ are multiple objectives to be optimized; and $G \subset R^n$ involves system constraints. The generalized definitions of Pareto optimality (Steuer, 1986; Yu, 1985) are given as follows (e.g. for minimization problem).

Definition 1 (*Pareto optimal solution*). A point, $x^* \in G$, is Pareto optimal solution if and only if there does not exit another $x \in G$, such that $f_i(x) \leqslant f_i(x^*)$ for all i, $(i = 1, \ldots, k)$ with strict inequality holding for at least one i.

Definition 2 (*Weak Pareto optimal solution*). A point, $x^* \in G$, is weak Pareto optimal solution if and only if there does not exit another $x \in G$, such that $f_i(x) < f_i(x^*)$ for all i, $(i = 1, \ldots, k)$.

The maximization problem has the similar definitions.

2.2. Fuzzy multiple objective optimization problem

In a fuzzy environment, according to Bellman and Zadeh (1970), Pal and Moitra (2003), the decision is often defined as follows:

$$\begin{aligned} &\text{Find}: \boldsymbol{x}, \\ &\text{so as to satisfy } f_i(\boldsymbol{x}) \begin{pmatrix} \lesssim \\ \cong \\ \gtrsim \end{pmatrix} f_i^*, \quad i = 1, \ldots, k, \\ &\text{subject to } x \in G, \end{aligned} \tag{2}$$

where f_i^* is the perspective goal value for the objective function $f_i(\boldsymbol{x})$; '$\lesssim$', '$\gtrsim$' and '$\cong$' express different fuzzy relations. For DM, the three types of fuzzy relations, respectively, denote that the ith fuzzy objective is approximately less than or equal to, approximately more than or equal to, and in the vicinity of f_i^*.

For multiple objective optimization problem in fuzzy environment, many researchers used the theory of fuzzy set. There are various kinds of membership functions such as linear, exponential, hyperbolic, hyperbolic-inverse, and piecewise-linear functions (Sakawa et al., 1987). The triangle-like membership functions are usually used for the objectives and their perspective goal values in literatures (Tiwari et al., 1987; Zimmermann, 1978). The corresponding membership functions are defined for three types of fuzzy relations in this paper.

For the fuzzy relation '$\lesssim$', the tolerant interval for the fuzzy objective is regarded as $(f_i^*, f_i^{\max})$. $f_i^{\max}$ is the tolerant limit for $f_i(\boldsymbol{x})$, as shown in Fig. 1.

Therefore the membership function is defined as

$$\mu_{f_i}(\boldsymbol{x}) = \begin{cases} 1 & f_i(\boldsymbol{x}) \leqslant f_i^*, \\ 1 - \frac{f_i(\boldsymbol{x}) - f_i^*}{f_i^{\max} - f_i^*} & f_i^* \leqslant f_i(\boldsymbol{x}) \leqslant f_i^{\max}, \\ 0 & f_i(\boldsymbol{x}) \geqslant f_i^{\max}. \end{cases} \tag{3}$$

The tolerant interval for '$\gtrsim$' which can be accepted by DM is $(f_i^{\min}, f_i^*)$. Fig. 2 illustrates the graph of this fuzzy relation.

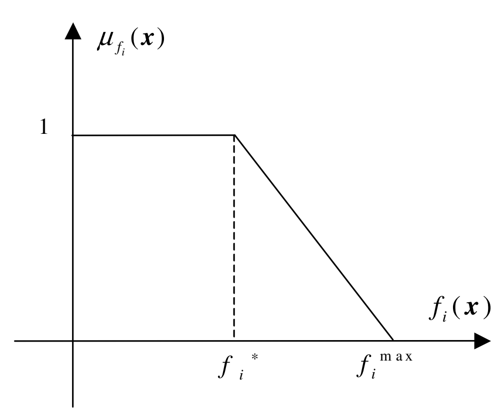

Fig. 1. Membership function $\mu_{f_i}(\boldsymbol{x})$ for fuzzy relation '$\lesssim$'.

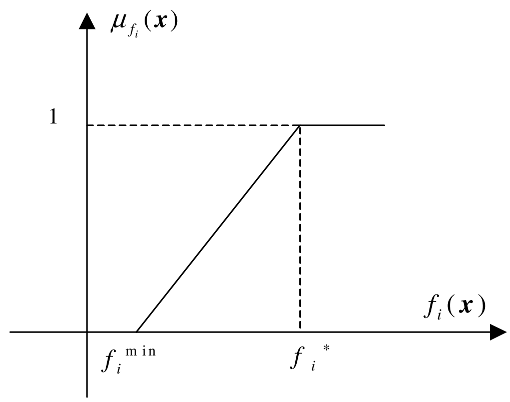

Fig. 2. Membership function $\mu_{f_i}(\boldsymbol{x})$ for fuzzy relation '$\gtrsim$'.

The membership function can take the form

$$\mu_{f_i}(\boldsymbol{x}) = \begin{cases} 1 & f_i(\boldsymbol{x}) \geqslant f_i^*, \\ 1 - \frac{f_i^* - f_i(\boldsymbol{x})}{f_i^* - f_i^{\min}} & f_i^{\min} \leqslant f_i(\boldsymbol{x}) \leqslant f_i^*, \\ 0 & f_i(\boldsymbol{x}) \leqslant f_i^{\min} \end{cases} \tag{4}$$

$(f_i^{\min}, f_i^{\max})$ is the tolerant interval for fuzzy relation '$\cong$' (see Fig. 3).

The membership function can be expressed as follows:

$$\mu_{f_i}(\boldsymbol{x}) = \begin{cases} 0 & f_i(\boldsymbol{x}) \geqslant f_i^{\max}, \\ 1 - \frac{f_i(\boldsymbol{x}) - f_i^*}{f_i^{\max} - f_i^*} & f_i^* \leqslant f_i(\boldsymbol{x}) \leqslant f_i^{\max}, \\ 1 & f_i(\boldsymbol{x}) = f_i^*, \\ 1 - \frac{f_i^* - f_i(\boldsymbol{x})}{f_i^* - f_i^{\min}} & f_i^{\min} \leqslant f_i(\boldsymbol{x}) \leqslant f_i^*, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

In MODM, however, the tolerant interval of the objective may be not legible. Hence, the payoff table is used to construct it. For example, the ideal solution of each objective under the system constraints for the fuzzy relation '$\lesssim$' is

$$f_i(\boldsymbol{x}^{i^*}) = \min_{x \in G} f_i(\boldsymbol{x}), \quad i = 1, \ldots, k \tag{6}$$

together with

$$f_{ij} = f_j(\boldsymbol{x}^{i^*}), \quad i,j = 1, \ldots, k. \tag{7}$$

Then the payoff table is formed as Table 1.

We can get the tolerant limit of every objective by the following equation:

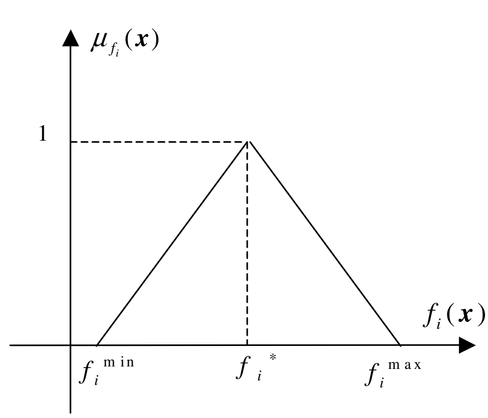

Fig. 3. Membership function $\mu_{f_i}(\boldsymbol{x})$ for fuzzy relation '$\cong$'.

Table 1
Payoff table of ideal solution

	f_1	f_2	f_3	$\cdots$	f_k
$\min f_1(\boldsymbol{x})$	$f_1(\boldsymbol{x}^{1^*})$	$f_2(\boldsymbol{x}^{1^*})$	$f_3(\boldsymbol{x}^{1^*})$		$f_k(\boldsymbol{x}^{1^*})$
$\min f_2(\boldsymbol{x})$	$f_1(\boldsymbol{x}^{2^*})$	$f_2(\boldsymbol{x}^{2^*})$	$f_3(\boldsymbol{x}^{2^*})$		$f_k(\boldsymbol{x}^{2^*})$
$\min f_3(\boldsymbol{x})$	$f_1(\boldsymbol{x}^{3^*})$	$f_2(\boldsymbol{x}^{3^*})$	$f_3(\boldsymbol{x}^{3^*})$		$f_k(\boldsymbol{x}^{3^*})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\min f_k(\boldsymbol{x})$	$f_1(\boldsymbol{x}^{k^*})$	$f_2(\boldsymbol{x}^{k^*})$	$f_3(\boldsymbol{x}^{k^*})$	$\cdots$	$f_k(\boldsymbol{x}^{k^*})$

$$f_j^{\max} = \max_{i=1,\ldots,k} f_{ij}, \quad j = 1, \ldots, k. \tag{8}$$

Having elicited the membership functions, if we introduce a general aggregation function

$$\mu_D(\boldsymbol{x}) = \mu_D(\mu_{f_1}(\boldsymbol{x}), \mu_{f_2}(\boldsymbol{x}), \ldots, \mu_{f_k}(\boldsymbol{x})) \tag{9}$$

fuzzy multiple objective optimization problem can be expressed by:

$$\begin{cases} \max & \mu_D(\boldsymbol{x}), \\ \text{s.t.} & x \in G. \end{cases} \tag{10}$$

For different aggregation operators, there are various formulations of fuzzy multiple objective optimization problem. Following Sakawa et al. (1987), the concepts of M-Pareto optimality are introduced here.

Definition 3 (*M-Pareto optimal solution*). A point, $x^* \in G$, is M-Pareto optimal solution if and only if there does not exit another $x \in G$, such that $\mu_{f_i}(\boldsymbol{x}) \geqslant \mu_{f_i}(\boldsymbol{x}^*)$ for all i, $(i = 1, \ldots, k)$ with strict inequality holding for at least one i.

Definition 4 (*Weak M-Pareto optimal solution*). A point, $x^* \in G$, is weak M-Pareto optimal solution if and only if there does not exit another $x \in G$, such that $\mu_{f_i}(\boldsymbol{x}) > \mu_{f_i}(\boldsymbol{x}^*)$ for all i, $(i = 1, \ldots, k)$.

3. Satisfying optimization method based on goal programming

3.1. Satisfying optimization reformulations

In the practical optimization and design, DM usually has an importance requirement for different objectives in multiple objective optimization problem as his or her preference. This can be expressed by weights. Tiwari et al. (1987) utilized the weighted additive formulation to realize it

$$\begin{cases} \max & V(\mu) = \sum_{i=1}^{k} w_i \mu_{f_i}(\boldsymbol{x}) \\ \text{s.t.} & \text{membership functions (3) or (4)} \\ & \mu_{f_i}(\boldsymbol{x}) \leqslant 1, \quad i = 1, \ldots, k, \\ & \boldsymbol{x} \in G. \end{cases} \tag{11}$$

This model optimizes each objective as much as possible.

However, DM maybe has a limited ability to specify explicit weight information. Thus the linguistic terms are generally used to denote the fuzzy importance of the objectives. They include "very important", "somewhat important", "important", "general", "unimportant", "somewhat unimportant", "very unimportant" seven terms. For example, the objective $f_j(\boldsymbol{x})$ is "very important", and $f_q(\boldsymbol{x})$ is "somewhat important", $j,q \in \{1, \ldots, k\}$, $j \neq q$.

Then it is impossible for the above model (11) to solve multiple objective optimization problem when DM only gives the linguistic terms information instead of the explicit weights. Therefore alternative fuzzy methods are presented. Chen and Tsai (2001) proposed the principle that the more important objective has the higher desirable achievement degree. That is

$$\mu_{f_i}(\boldsymbol{x}) \geqslant \mu_{f_i}^*, \quad i = 1, \ldots, k, \tag{12}$$

where $\mu^*_{f_i}$ is the desirable satisfying degree of the objective $f_i(\boldsymbol{x})$, and is given by DM in advance. They incorporated inequalities (12) into the fuzzy programming (11) as additional constraints, which simplifies original problem. The optimization model is reformulated:

$$\begin{cases} \max & V(\mu) = \sum_{i=1}^{k} \mu_{f_i}(\boldsymbol{x}) \\ \text{s.t.} & \text{membership functions (3) or (4)}, \\ & \mu_{f_i}(\boldsymbol{x}) \geqslant \mu^*_{f_i}, \quad i = 1, \ldots, k, \\ & \mu_{f_i}(\boldsymbol{x}) \leqslant 1, \quad \boldsymbol{x} \in G. \end{cases} \tag{13}$$

Unfortunately, the additional constraints (12) may be too strict for solving the reformulation (13). Therefore, there may be no satisfying even feasible solution if the DM demands a high achievement for a special fuzzy objective under linguistic terms. The numerical examples about efficiency of our method in Section 5 will show that this case possibly exists. Simultaneously, it is hard to specify the desirable satisfying degrees of all the objectives in advance.

For priority requirement, the strict order between satisfying degrees requires the higher priority achieving the higher satisfying degree (Chen and Tsai, 2001; Hu and Li, 2006; Li et al., 2004). In Li et al. (2004), they made use of varying-domain optimization method to solve multiple objective optimization problem with preemptive priorities. In their model, the priority relations are realized by the following inequalities:

$$\begin{cases} \alpha\beta_i - \beta_i \leqslant f_i(\boldsymbol{x}) \leqslant \beta_i - \alpha\beta_i, \\ 0 \leqslant \beta_i \leqslant 1, \quad i = 1, \ldots, k, \\ \beta_j - \beta_{j-1} \leqslant \gamma, \quad j = 2, \ldots, k. \end{cases} \tag{14}$$

The priorities are expressed by introducing a new decision variable γ and vary-domain variables β_i ($i = 1, \ldots, k$). $[\alpha\beta_i - \beta_i, \beta_i - \alpha\beta_i]$ is the varying-domain of the objective $f_i(\boldsymbol{x})$. The constraints, $\beta_j - \beta_{j-1} \leqslant \gamma$, ($j = 2, \ldots, k$), denote the priority difference between the objectives. The priority requirement is realized by optimizing γ. The smaller γ is, the more obvious the priority structure is. It means that the difference between the domains of the objectives is larger.

Since the priority requirement is the limit of importance, the requirement that an objective with higher priority has a higher satisfying degree is not necessary in this paper. Thus, we hold that a more important objective has a higher desirable achievement degree, which is looser than the priority requirement. The optimal result satisfying our principle only owns the order of desirable satisfying degree although the real individual satisfying degree may violate it. Therefore, we regard those desirable satisfying degrees as optimization variables instead of deterministic values.

For the multiple objective optimization problem, it is ordinarily impossible to give the specific quantities of importance between the objectives when there is imprecise preference from DM. Thus, the additional constraints need to be incorporated to make the solution exist and satisfy the objectives and preference requirement of DM. In this paper, the different importance is expressed using linguistic terms. The importance difference between the corresponding objectives can be realized by comparison between their desirable satisfying degrees. Such as the objective $f_j(\boldsymbol{x})$is "very important", and $f_q(\boldsymbol{x})$ is "somewhat important", $j,q \in \{1, \ldots, k\}$, $j \neq q$. The crisp comparison relation is expressed by the following formulation

$$\mu^*_{f_q} \leqslant \mu^*_{f_j}, \tag{15}$$

where $\mu^*_{f_j}$, $\mu^*_{f_q}$ are, respectively, the desirable satisfying degrees of $f_j(\boldsymbol{x})$ and $f_q(\boldsymbol{x})$. In order to guarantee the feasible and satisfying solution, the ranking strategy in Li et al. (2004) is used in this paper to compare the importance among these objectives. The similar constraints to $\beta_j - \beta_{j-1} \leqslant \gamma$ in (14) and the decision variable $\gamma(-1 \leqslant \gamma \leqslant 1)$ are also incorporated into (15) to form the new comparison inequality in this paper. Then

$$\mu^*_{f_q} - \mu^*_{f_j} \leqslant \gamma, \tag{16}$$

where γ is called importance difference variable in this paper. By means of (16), the desirable satisfying degrees of different important objectives are divided into various levels.

In this paper, our method is introduced to reformulate this problem. The trade-off between optimization and importance is realized in terms of the idea of satisfying optimization (Goodrich et al., 1998) in order to guarantee feasible solution. And the computation is correspondingly decreased. For optimizing every objective under the aspiration value as much as possible, we refer to the additive model in Chen and Tsai (2001) and Tiwari et al. (1987). Here, all of the desirable satisfying degrees are decision variables. Thus the goal of our new model is to maximize the sum of all desirable satisfying degrees. This not only maximizes each desirable satisfying degree, but also acquires the maximum individual satisfying degree under the fuzzy importance. The optimization model is formulated as follows:

$$\begin{cases} \max & \sum_{i=1}^{k} \mu^*_{f_i}/k - \lambda \cdot \gamma \\ \text{s.t.} & \text{membership functions (3) or (4)}, \\ & \mu_{f_i}(\boldsymbol{x}) \geqslant \mu^*_{f_i}, \quad i = 1, \ldots, k, \\ & \mu^*_{f_q} - \mu^*_{f_j} \leqslant \gamma, \quad j,q \in \{1, \ldots, k\}, \ j \neq q, \\ & \mu_{f_i} \leqslant 1, \\ & \boldsymbol{x} \in G. \end{cases} \tag{17}$$

In optimization model (17), the aim of minimizing γ is to obtain the order of desirable satisfying degrees as far as possible. Since $\mu^*_{f_j}$ is located in the interval [0,1], γ belongs to [-1,1]. If $\gamma > 0$, the solution does not satisfy the fuzzy importance. On the contrary, the satisfying solution is acquired if $\gamma \leqslant 0$ and the results conform to the preference extent.

Although the optimization model (17) is efficient, it is still unable to deal with the fuzzy relation '$\cong$'. The membership function constraints $\mu_{f_i}(\boldsymbol{x}) \leqslant 1$ will lead to that there is only one feasible solution for this fuzzy relation. That is just the perspective goal for this objective. The limitation of this study is that the additive model is only used to solve optimization problem with '$\tilde{\leqslant}$' and '$\tilde{\geqslant}$'. Therefore, we propose the satisfying optimization method based on goal programming to solve the multiple objective optimization problem with three types of fuzzy relations. Firstly, the problem with the inequity fuzzy relations is taken as an example to explain the new optimization model. Then the generalization formulation for three types of fuzzy relations is given.

Regarding the inequity fuzzy relations '$\tilde{\leqslant}$' and '$\tilde{\geqslant}$', the objectives in (3) and (4) can, respectively, be transformed by means of goal programming. For '$\tilde{\leqslant}$', supposing that the values of objectives are in their tolerant ranges, the new formulation is

$$f_i(\boldsymbol{x}) - p_i = f_i^*, \quad i = 1, \ldots, k. \tag{18}$$

The membership function $\mu_{f_i}(\boldsymbol{x})$ is converted into

$$\mu_{f_i}(\boldsymbol{x}) = 1 - p_i/(f_i^{\max} - f_i^*) \tag{19}$$

where $p_i(p_i \geqslant 0)$ is positive deviational variable.

And the formulation for '$\tilde{\geqslant}$' is

$$f_i(\boldsymbol{x}) + n_i = f_i^*, \quad i = 1, \ldots, k. \tag{20}$$

The corresponding conclusion to the membership function $\mu_{f_i}(\boldsymbol{x})$ is

$$\mu_{f_i}(\boldsymbol{x}) = 1 - n_i/(f_i^* - f_i^{\min}) \tag{21}$$

and $n_i(n_i \geqslant 0)$ is negative deviational variable.

Consequently, for the multiple objective optimization problem with fuzzy relation '$\tilde{\lessgtr}$' and '$\tilde{\gtrless}$' and fuzzy importance, the following symbols, respectively, express the sets including the objetives with the same linguistic term:

S_{vi}: "very important"
S_{si}: "somewhat important"
S_i: "important"
S_g: "general"
S_u: "unimportant"
S_{su}: "somewhat unimportant"
S_{vu}: "very unimportant"

where the intersection between the different sets is empty. For a multiple objective optimization with inequity fuzzy relations, if $f_j(\boldsymbol{x})$ is "very important", $f_q(\boldsymbol{x})$ is "somewhat important", then the problem can be described as follows

Find: $\boldsymbol{x}$
so as to satisfy

$$\begin{aligned} &f_i(\boldsymbol{x})\tilde{\lessgtr} f_i^* \quad i=1,\ldots,k_1,\\ &f_l(\boldsymbol{x})\tilde{\gtrless} f_l^* \quad l=k_1+1,\ldots,k, \end{aligned} \tag{22}$$

subject to

$$f_j(\boldsymbol{x})\in S_{vi},\quad f_q(\boldsymbol{x})\in S_{si},\quad S_{vi},S_{si}\subset\{1,\ldots,k\},\quad S_{vi}\cap S_{si}=\varnothing \quad \text{and } x\in G.$$

We formulate the optimization model

$$\begin{cases} \max & \left(\sum_{i=1}^{k_1}\mu_{f_i}^* + \sum_{l=k_1+1}^{k}\mu_{f_l}^*\right)\Big/ k - \lambda\cdot\gamma \\ \text{s.t.} & f_i(\boldsymbol{x}) + n_i - p_i = f_i^*,\quad i=1,\ldots,k_1,\\ & f_l(\boldsymbol{x}) + n_l - p_l = f_l^*,\quad l=k_1+1,\ldots,k,\\ & 1-p_i/(f_i^{\max}-f_i^*) \geqslant \mu_{f_i}^*,\\ & 1-n_l/(f_l^*-f_l^{\min}) \geqslant \mu_{f_l}^*,\\ & \mu_{f_q}^*-\mu_{f_j}^* \leqslant \gamma,\\ & f_j(\boldsymbol{x})\in S_{vi},\quad f_q(\boldsymbol{x})\in S_{si},\\ & S_{vi},S_{si}\subset\{1,\ldots,k\},\quad S_{vi}\cap S_{si}=\varnothing,\\ & n_l\leqslant f_l^*-f_l^{\min},\quad p_i\leqslant f_i^{\max}-f_i^*,\\ & n_i,p_i,n_l,p_l,\mu_{f_i}^*,\mu_{f_l}^*,\mu_{f_j}^*,\mu_{f_q}^* \geqslant 0,\\ & n_i\cdot p_i=0,\quad n_l\cdot p_l=0,\\ & \boldsymbol{x}\in G. \end{cases} \tag{23}$$

Similarly, the transformation for the equity fuzzy relation '$\cong$' is implemented. The objective is reformulated as

$$f_i(\boldsymbol{x}) + n_i - p_i = f_i^*,\quad i=1,\ldots,k. \tag{24}$$

By means of deviational variables n_i,p_i, the membership function $\mu_{f_i}(\boldsymbol{x})$ is expressed as

$$\mu_{f_i}(\boldsymbol{x}) = \begin{cases} 1-p_i/(f_i^{\max}-f_i^*) & p_i>0,\\ 1-n_i/(f_i^*-f_i^{\min}) & n_i>0,\\ 1 & p_i=n_i=0. \end{cases} \tag{25}$$

Then for the following hybrid multiple objective optimization problem, i.e. including inequality and equality fuzzy relations

Find: $\boldsymbol{x}$
so as to satisfy

$$\begin{aligned} &f_i(\boldsymbol{x})\tilde{\lessgtr} f_i^* \quad i=1,\ldots,k_1,\\ &f_t(\boldsymbol{x})\tilde{\gtrless} f_t^* \quad t=k_1+1,\ldots,k_2,\\ &f_s(\boldsymbol{x})\cong f_s^* \quad s=k_2+1,\ldots,k, \end{aligned} \tag{26}$$

subject to

$$f_j(\boldsymbol{x})\in S_{vi},\quad f_q(\boldsymbol{x})\in S_{si},\quad S_{vi},S_{si}\subset\{1,\ldots,k\},\quad S_{vi}\cap S_{si}=\varnothing, \text{and } \boldsymbol{x}\in G$$

according to the optimization model (23) for the inequality fuzzy relations, the generalization optimization formulation can be proposed as follows:

$$\begin{cases} \max & \left(\sum_{i=1}^{k_1}\mu_{f_i}^* + \sum_{t=k_1+1}^{k_2}\mu_{f_t}^* + \sum_{s=k_2+1}^{k}\mu_{f_s}^*\right)\Big/ k - \lambda\cdot\gamma \\ \text{s.t.} & f_i(\boldsymbol{x}) + n_i - p_i = f_i^*,\quad i=1,\ldots,k_1,\\ & f_t(\boldsymbol{x}) + n_t - p_t = f_t^*,\quad t=k_1+1,\ldots,k_2,\\ & f_s(\boldsymbol{x}) + n_s - p_s = f_s^*,\quad s=k_2+1,\ldots,k,\\ & 1-p_i/(f_i^{\max}-f_i^*) \geqslant \mu_{f_i}^*,\\ & 1-n_t/(f_t^*-f_t^{\min}) \geqslant \mu_{f_t}^*,\\ & 1-(n_s/(f_s^*-f_s^{\min})+p_s/(f_s^{\max}-f_s^*)) \geqslant \mu_{f_s}^*\\ & \mu_{f_q}^*-\mu_{f_j}^* \leqslant \gamma\\ & f_j(\boldsymbol{x})\in S_{vi},\quad f_q(\boldsymbol{x})\in S_{si},\\ & S_{vi},S_{si}\subset\{1,\ldots,k\},\quad S_{vi}\cap S_{si}=\varnothing,\\ & n_t\leqslant f_t^*-f_t^{\min},\quad p_i\leqslant f_i^{\max}-f_i^*,\\ & n_s\leqslant f_s^*-f_s^{\min},\quad p_s\leqslant f_s^{\max}-f_s^*,\\ & n_i,p_i,n_t,p_t,n_s,p_s,\mu_{f_i}^*,\mu_{f_t}^*,\mu_{f_s}^*,\mu_{f_j}^*,\mu_{f_q}^* \geqslant 0,\\ & n_i\cdot p_i=0,\quad n_t\cdot p_t=0,\quad n_s\cdot p_s=0,\quad \boldsymbol{x}\in G. \end{cases} \tag{27}$$

Model (27) can be taken as the general formulation to solve the multiple objective optimization problem with any type of fuzzy relation.

Theorem. *For any $\lambda(\lambda>0)$, there must, respectively, exist an optimal solution of the programming model (27) for any type of fuzzy relation when G is non-empty, and $f_i^{\max}-f_i^*, f_t^*-f_t^{\min}, f_s^*-f_s^{\min}, f_s^{\max}-f_s^*$ are always positive.*

Proof. The constraints $\mu_{f_q}^*-\mu_{f_j}^*\leqslant\gamma$ of (27) can be indexed as $(\mu_{f_q}^*-\mu_{f_j}^*)_{ii}\leqslant\gamma$, $(ii=1,\ldots,IM)$. IM is the number of comparison inequalities about importance. When λ is given, they will be classified into active constraints and inactive constraints during solving. Then all these constraints can be expressed as the following formulation

$$\gamma = \max_{ii}[(\mu_{f_q}^*-\mu_{f_j}^*)_{ii}]. \tag{28}$$

Therefore, the programming model (27) is reformulated as follows:

$$\begin{cases} \max & \left(\left(\sum_{i=1}^{k_1}\mu_{f_i}^* + \sum_{t=k_1+1}^{k_2}\mu_{f_t}^* + \sum_{s=k_2+1}^{k}\mu_{f_s}^*\right)\Big/ k - \lambda \times(\max_{ii}[(\mu_{f_q}^*-\mu_{f_j}^*)_{ii}])\right) \\ \text{s.t.} & f_i(\boldsymbol{x}) + n_i - p_i = f_i^*,\quad i=1,\ldots,k_1,\\ & f_t(\boldsymbol{x}) + n_t - p_t = f_t^*,\quad t=k_1+1,\ldots,k_2,\\ & f_s(\boldsymbol{x}) + n_s - p_s = f_s^*,\quad s=k_2+1,\ldots,k,\\ & 1-p_i/(f_i^{\max}-f_i^*) \geqslant \mu_{f_i}^*,\\ & 1-n_t/(f_t^*-f_t^{\min}) \geqslant \mu_{f_t}^*,\\ & 1-(n_s/(f_s^*-f_s^{\min})+p_s/(f_s^{\max}-f_s^*)) \geqslant \mu_{f_s}^*,\\ & n_t\leqslant f_t^*-f_t^{\min},\quad p_i\leqslant f_i^{\max}-f_i^*,\\ & n_s\leqslant f_s^*-f_s^{\min},\quad p_s\leqslant f_s^{\max}-f_s^*,\\ & n_i,p_i,n_t,p_t,n_s,p_s,\mu_{f_i}^*,\mu_{f_t}^*,\mu_{f_s}^*,\mu_{f_j}^*,\mu_{f_q}^* \geqslant 0,\\ & n_i\cdot p_i=0,\quad n_t\cdot p_t=0,\quad n_s\cdot p_s=0,\\ & \boldsymbol{x}\in G. \end{cases} \tag{29}$$

It is obvious that the two programming models (27) and (29) are equivalent. For the programming model (29), the constraints field is the basic sets about all objective functions. Thus the programming is also feasible when G is non-empty (i.e. $x \in G$ is feasible), and $f_i^{\max} - f_i^*$, $f_t^* - f_t^{\min}$, $f_s^* - f_s^{\min}$, $f_s^{\max} - f_s^*$ are always positive. It can be seen that there must exist an optimal solution of the programming model (27) if the above conditions are satisfied. □

3.2. Parameter λ analysis

3.2.1. Alteration of λ

The optimization objective function of model (27) consists of two parts: the sum of desirable achievement degrees and the importance difference. This shows that the satisfying result is compromise between optimization and importance requirement through interaction with DM. For the programming model (27), the single optimization objective function can be rewritten as follows

$$\max h_1(\zeta) + \lambda \cdot h_2(\zeta), \tag{30}$$

where $h_1(\zeta)$ denotes the sum of desirable achievement degrees, i.e. $\left(\sum_{i=1}^{k_1} \mu_{f_i}^* + \sum_{t=k_1+1}^{k_2} \mu_{f_t}^* + \sum_{s=k_2+1}^{k} \mu_{f_s}^*\right)/k$; $h_2(\zeta)$ is the importance and equal to $(-\gamma)$; ζ is the decision vector including $\boldsymbol{x}$, $\boldsymbol{n}$, $\boldsymbol{p}$, γ. Then parameter λ acts as the weight. The normalized weights ϖ_1 and ϖ_2 can be, respectively, defined as $1/(1+\lambda)$ and $\lambda/(1+\lambda)$. When $\lambda \to 0$, $\varpi_1 \to 1$. And the objective $h_1(\zeta)$ will attain maximum regardless of $h_2(\zeta)$ under the goal constraints. As λ increases, i.e. ϖ_2 increases, the $h_2(\zeta)$ will be emphasized. According to the definitions of the two objectives, we can see that both of the sum of desirable satisfying degrees will decrease and the importance difference γ will become negative as λ increases. When $\lambda \to \infty$, $\varpi_2 \to 1$. The objective $h_1(\zeta)$ will be ignored. The maximization of $h_2(\zeta)$ is obtained by solving the following programming

$$\max h_2(\zeta). \tag{31}$$

In a word, by regulating parameter λ, DM can acquire his or her desirable solution. When DM emphasizes the maximization of the desirable achievement degrees, he can reduce λ, and vice versa.

3.2.2. Minimum λ^* for limit case

Simultaneously, some of the desirable satisfying degrees will get zero and the solution will remain identical when $\lambda > \lambda^*$. This is the limit case of distribution of importance. λ^* is the minimum parameter which can lead to this case. In this paper, we present the algorithm to find λ^*. As the parametric programming, the sensitivity analysis of linear programming is used to realize it. For the programming model (27), the following auxiliary linear programming is adapted to acquire the minimum λ.

$$\begin{cases} \max & \left(\sum_{i=1}^{k_1} \mu_{f_i}^* + \sum_{t=k_1+1}^{k_2} \mu_{f_t}^* + \sum_{s=k_2+1}^{k} \mu_{f_s}^*\right) \Big/ k - (0+\lambda)\cdot\gamma \\ \text{s.t.} & 1 - p_i^*/(f_i^{\max} - f_i^*) \geqslant \mu_{f_i}^*, \quad i = 1,\ldots,k_1, \\ & 1 - n_t^*/(f_t^* - f_t^{\min}) \geqslant \mu_{f_t}^*, \quad t = k_1+1,\ldots,k_2, \\ & 1 - (n_s^*/(f_s^* - f_s^{\min}) + p_s^*/(f_s^{\max} - f_s^*)) \geqslant \mu_{f_s}^*, \\ & s = k_2+1,\ldots,k, \\ & \mu_{f_q}^* - \mu_{f_j}^* \leqslant \gamma, \\ & f_j(\boldsymbol{x}) \in S_{vi}, \quad f_q(\boldsymbol{x}) \in S_{si}, \\ & S_{vi}, S_{si} \subset \{1,\ldots,k\}, \quad S_{vi} \cap S_{si} = \varnothing. \end{cases} \tag{32}$$

The linear programming is solved by means of the simplex method. Let μ^o express the sum of desirable satisfying degrees.

Algorithm.

Step 1 Initially transform the optimization objective of the programming (27) into "max$(-\gamma)$", and acquire the solution x^*, n^*, p^*, γ^*.
Step 2 Formulate auxiliary programming (32) with the above x^*, n^*, p^*, and solve it by simplex method when $\lambda = 0$.
Step 3 According the obtained simplex table, determine the different optimum results with the corresponding various λ by means of sensitivity analysis. Finally acquire the minimum λ^o when the maximum value of the objective is unaltered.
Step 4 Substitute λ^o into (27) and judge: if the result γ^o is identical with that in step 1, λ^o is the final solution; if the case does not exist, λ^o is not the solution, and go to the next step.
Step 5 Using the above result, solve the following equality

$$\mu^o/k - \lambda \cdot \gamma^o = \mu^*/k - \lambda \cdot \gamma^*. \tag{33}$$

The solution is λ^*. Solve (27) with λ^*, if the result is equal to γ^*, λ^* is considered as the final solution and algorithm stops; or else, increase λ^* properly and express it as λ^o, go back to step 4.

3.3. Numerical test for M-Pareto optimality

The corresponding model testing the M-Pareto optimality of $\tilde{\boldsymbol{x}}$ for the programming model (27) is performed by means of solving the following model:

$$\begin{cases} \max & \sum_{i=1}^{k_1} \varepsilon_i + \sum_{t=k_1+1}^{k_2} \varepsilon_t + \sum_{s=k_2+1}^{k} \varepsilon_s \\ \text{s.t.} & f_i(\boldsymbol{x}) + n_i - p_i = f_i^*, \quad i = 1,\ldots,k_1, \\ & f_t(\boldsymbol{x}) + n_t - p_t = f_t^*, \quad t = k_1+1,\ldots,k_2, \\ & f_s(\boldsymbol{x}) + n_s - p_s = f_s^*, \quad s = k_2+1,\ldots,k, \\ & p_i/(f_i^{\max} - f_i^*) + \varepsilon_i = \tilde{p}_i/(f_i^{\max} - f_i^*), \\ & n_t/(f_t^* - f_t^{\min}) + \varepsilon_t = \tilde{n}_t/(f_t^* - f_t^{\min}), \\ & n_s/(f_s^* - f_s^{\min}) + p_s/(f_s^{\max} - f_s^*) + \varepsilon_s \\ & \quad = \tilde{n}_s/(f_s^* - f_s^{\min}) + \tilde{p}_s/(f_s^{\max} - f_s^*), \\ & n_i, p_i, n_t, p_t, n_s, p_s \geqslant 0, n_i \cdot p_i = 0, n_t \cdot p_t = 0, n_s \cdot p_s = 0, \\ & \varepsilon_i \geqslant 0, \quad \varepsilon_t \geqslant 0, \quad \varepsilon_s \geqslant 0, \\ & x \in G, \end{cases} \tag{34}$$

where $(\varepsilon_i, \varepsilon_t, \varepsilon_s)$ are the error variables, $(\tilde{\boldsymbol{n}}, \tilde{\boldsymbol{p}})$ are the deviational variables for $\tilde{x}$, and (f_i^*, f_t^*, f_s^*) are the perspective goal value presented in (2). Let $\bar{x}$, $(\bar{\varepsilon}_i, \bar{\varepsilon}_t, \bar{\varepsilon}_s)$ be an optimal solution of (34). It is observed that if all $(\bar{\varepsilon}_i, \bar{\varepsilon}_t, \bar{\varepsilon}_s)$ are zero, $\tilde{\boldsymbol{x}}$ is a M-Pareto optimal solution. And it is also a Pareto optimal solution when the tolerances for positive and negative deviational variables are identical; if at least one $\bar{\varepsilon}_i$, $\bar{\varepsilon}_t$ or $\bar{\varepsilon}_s$ is not zero, $\tilde{\boldsymbol{x}}$ becomes a M-Pareto optimal solution and Pareto optimal solution. Note: (34) acts as goal programming, then the constraint "$n_i \cdot p_i = 0$" can be ignored during solving and the result will not be affected.

4. Optimization algorithm

According to the proposed satisfying optimization method based on goal programming, the following algorithm for fuzzy multiple objective optimization under fuzzy importance is given as follows

Step 1. Formulate the proper optimization model according to the fuzzy relations and preference of DM expressed by the linguistic terms in original optimization problem.

Step 2. Initially solve the reformulated optimization problem with a small λ.

Step 3. Judge: if there is the importance difference variable $\gamma > 0$, go to next step. If $\gamma \leqslant 0$ but not satisfy DM, go to next step, too; otherwise optimization stop, and the satisfying solution is acquired.

Step 4. Increase λ, and solve the reformulation again, then go back to step 3 and continue.

5. Numerical examples

We demonstrate for the efficiency, flexibility and sensitivity of the proposed method by the following examples and their alteration in this paper.

Example 1 (*Chen and Tsai, 2001; Li et al., 2004; Tiwari et al., 1987*). Find $x(x_1,x_2,x_3,x_4)$ to satisfy

$$\begin{cases} f_1(\boldsymbol{x}): & 4x_1 + 2x_2 + 8x_3 + x_4 \tilde{\leqslant} 35 \\ f_2(\boldsymbol{x}): & 4x_1 + 7x_2 + 6x_3 + 2x_4 \tilde{\geqslant} 100 \\ f_3(\boldsymbol{x}): & x_1 - 6x_2 + 5x_3 + 10x_4 \tilde{\geqslant} 120 \\ f_4(\boldsymbol{x}): & 5x_1 + 3x_2 + 2x_4 \tilde{\geqslant} 70 \\ f_5(\boldsymbol{x}): & 4x_1 + 4x_2 + 4x_3 \tilde{\geqslant} 40 \end{cases} \tag{35}$$

subject to

$$\begin{cases} 7x_1 + 5x_2 + 3x_3 + 2x_4 \leqslant 98 \\ 7x_1 + x_2 + 6x_3 + 6x_4 \leqslant 117 \\ x_1 + x_2 + 2x_3 + 6x_4 \leqslant 130 \\ 9x_1 + x_2 + 6x_4 \leqslant 105 \\ x_i \geqslant 0, i = 1, \ldots, 4 \end{cases} \tag{36}$$

The tolerant limits of the five fuzzy objectives are (55, 40, 70, 30, 10), respectively. The fuzzy importance requirement is: $f_1(\boldsymbol{x})$ and $f_5(\boldsymbol{x})$ are "very important"; $f_2(\boldsymbol{x})$ is "somewhat important"; $f_4(\boldsymbol{x})$ is "important"; $f_3(\boldsymbol{x})$ is "general".

Firstly, according to the fuzzy relations of the objectives, the optimization model is reformulated as follows:

$$\begin{cases} \max & \sum_{i=1}^{5} \mu_{f_i}^*/5 - \lambda \cdot \gamma \\ \text{s.t.} & 4x_1 + 2x_2 + 8x_3 + x_4 + n_1 - p_1 = 35, \\ & 4x_1 + 7x_2 + 6x_3 + 2x_4 + n_2 - p_2 = 100, \\ & x_1 - 6x_2 + 5x_3 + 10x_4 + n_3 - p_3 = 120, \\ & 5x_1 + 3x_2 + 2x_4 + n_4 - p_4 = 70, \\ & 4x_1 + 4x_2 + 4x_3 + n_5 - p_5 = 40, \\ & 1 - p_1/20 \geqslant \mu_{f_1}^*, \\ & 1 - n_2/60 \geqslant \mu_{f_2}^*, \\ & 1 - n_3/50 \geqslant \mu_{f_3}^*, \\ & 1 - n_4/40 \geqslant \mu_{f_4}^*, \\ & 1 - n_5/30 \geqslant \mu_{f_5}^*, \\ & \mu_{f_2}^* - \mu_{f_1}^* \leqslant \gamma, \\ & \mu_{f_2}^* - \mu_{f_5}^* \leqslant \gamma, \\ & \mu_{f_4}^* - \mu_{f_2}^* \leqslant \gamma, \\ & \mu_{f_3}^* - \mu_{f_4}^* \leqslant \gamma, \\ & p_1 \leqslant 20, \quad n_2 \leqslant 60, \quad n_3 \leqslant 50, \quad n_4 \leqslant 40, \quad n_5 \leqslant 30, \\ & n_i, p_i, \mu_{f_i}^* \geqslant 0, \quad n_i \cdot p_i = 0, \quad i = 1, \ldots, 5, \\ & \text{system constraints (36)} \end{cases} \tag{37}$$

Solving (37) by LINGO, and then testing the M-Pareto optimality of the solutions. The different results according to different λ are listed in Table 2.

Form Table 2, the sum of desirable satisfying degrees and variable γ decrease monotonously with the increment of λ. The order of desirable satisfying degrees is not consistent with the given relative importance under $\gamma > 0$. Moreover we can see the solution may remain invariable in the interval of λ. DM considers the solution (0.0000, 9.8148, 0.0000, 15.8642) as his or her preferred one when $\gamma \in (-0.0980, -0.1267)$. The desirable satisfying degrees of all objectives conform to DM's linguistic terms. The optimization result of each objective satisfies DM.

5.1. Minimum λ^*

When $\gamma = -0.3333$, the $\mu_{f_3}^*$ will become 0 and the corresponding solution is stable. The auxiliary linear programming like (32) and its algorithm are used to acquire the minimum λ^*.

Firstly, when $\lambda \to \infty$, solve the programming (37) and get the following result:

$$\begin{cases} \boldsymbol{x}^* = (0.0000, 10.0000, 0.0000, 15.0000), \\ \boldsymbol{n}^* = (0.0000, 0.0000, 30.0000, 10.0000, 0.0000), \\ \boldsymbol{p}^* = (0.0000, 0.0000, 0.0000, 0.0000, 0.0000), \\ \gamma^* = -0.3333. \end{cases}$$

According to the above solution, the auxiliary linear programming model is formulated as

$$\begin{cases} \max & \sum_{i=1}^{5} \mu_{f_i}^*/5 - \lambda \cdot \gamma \\ \text{s.t.} & 1 - p_1^*/20 \geqslant \mu_{f_1}^*, \\ & 1 - n_2^*/60 \geqslant \mu_{f_2}^*, \\ & 1 - n_3^*/50 \geqslant \mu_{f_3}^*, \\ & 1 - n_4^*/40 \geqslant \mu_{f_4}^*, \\ & 1 - n_5^*/30 \geqslant \mu_{f_5}^*, \\ & \mu_{f_2}^* - \mu_{f_1}^* \leqslant \gamma, \\ & \mu_{f_2}^* - \mu_{f_5}^* \leqslant \gamma, \\ & \mu_{f_4}^* - \mu_{f_2}^* \leqslant \gamma, \\ & \mu_{f_3}^* - \mu_{f_4}^* \leqslant \gamma, \\ & \gamma \leqslant 0. \end{cases} \tag{38}$$

Let $\lambda = 0$, the above auxiliary programming is solved by simplex method. According to the sensitivity analysis of linear programming, the solution is optimal and identical when all the reduced costs are greater than or equal to 0. Therefore the solution remains constant and γ attains minimum when $\lambda > 1.2$ after several pivot iterations of simplex method.

Substitute $\lambda = 1.2$ into (37) and solve it. The result $\gamma^o = -0.3333$ shows that this is the final solution. That means the sum of desirable satisfying degrees and the importance difference will remain constant if $\lambda > 1.2$. Moreover some of the satisfying degrees will become zero, then $\lambda^* = 1.2$.

5.2. Efficiency

For the original problem (35) and (36), there exists the feasible solution regardless of preference requirement. The following optimal solution is acquired through the additive model (Tiwari et al., 1987).

$$\begin{cases} \boldsymbol{x}^* = (0.0000, 9.7500, 0.0000, 15.8750), \\ \boldsymbol{\mu}^* = (0.9813, 1.0000, 0.6050, 0.7750, 0.9667), \\ \boldsymbol{f}^* = (37.3750, 100.0000, 100.2500, 61.0000, 39.0000). \end{cases}$$

However, there might be no feasible solution if the preference is required and the Chen and Tsai's model (13) is used. Suppose DM gives the following preference: $f_1(\boldsymbol{x})$ and $f_5(\boldsymbol{x})$ are "very important";

Table 2
Optimization results with different λ for Example 1

λ	Sum of desirable satisfying degrees	Importance difference	Individual desirable satisfying degree	Individual satisfying degree	Solution
0.05	4.3279	0.0333	(0.9813, 1.0000, 0.6050, 0.7750, 0.9667)	(0.9813, 1.0000, 0.6050, 0.7750, 0.9667)	(0.0000, 9.7500, 0.0000, 15.8750)
0.3	4.2023	−0.0980	(0.9753, 0.8773, 0.5951, 0.7793, 0.9753)	(0.9753, 1.0000, 0.5951, 0.7793, 0.9753)	(0.0000, 9.8148, 0.0000, 15.8642)
0.8	4.1161	−0.1267	(0.9753, 0.8486, 0.5951, 0.7218, 0.9753)	(0.9753, 1.0000, 0.5951, 0.7793, 0.9753)	(0.0000, 9.8148, 0.0000, 15.8642)
1.0	3.8000	−0.2000	(1.0000, 0.8000, 0.4000, 0.6000, 1.0000)	(1.0000, 1.0000, 0.4000, 0.7500, 1.0000)	(0.0000, 10.0000, 0.0000, 15.0000)
1.5	3.0000	−0.3333	(1.0000, 0.6667, 0.0000, 0.3333, 1.0000)	(1.0000, 1.0000, 0.4000, 0.7500, 1.0000)	(0.0000, 10.0000, 0.0000, 15.0000)

$f_2(\boldsymbol{x})$ is "somewhat important"; $f_4(\boldsymbol{x})$ is "general"; $f_3(\boldsymbol{x})$ is "unimportant". For example, if DM determines the desirable satisfying degrees of the objectives as $(1.0, 0.9, 0.65, 0.7, 0.95)$ in advance, the optimization problem becomes infeasible. Moreover, he or she does not know which desirable satisfying degree leads to this case. Nevertheless, there is always solution by our method even though the fuzzy importance is not satisfied in this paper. We can know this from the parameter $\gamma > 0$. DM may think that the result for $\gamma = -0.2000$ in Table 2 is satisfying.

In addition, the solutions of our approach are all M-Pareto optimal. This can be seen from the comparions with those in Aköz and Petrovic (2007). Firstly, solve this problem with originally given fuzzy importance by means of their approach using fuzzy binary relations "$\widetilde{R}(1,2) = \widetilde{R}(5,2) = \widetilde{R}(2,4) = \widetilde{R}(4,3) = \widetilde{R}_2$". The results are acquired as in the Table 3.

Where the parameter λ is different from ours, the optimization procedure with its changing has the contrary monotonicity.

It is known that Akoz and Petrovic's method is systemic and reasonable. Unfortunately there are the inferior solutions when $\lambda \leqslant 0.51$ according to the optimization results in Tables 2 and 3. Therefore only the first solution in Table 3 is M-Pareto optimal and applicable. However beside this, there are still others solutions that are able to be used in our results. Alternatively, the fuzzy binary relation between $f_1(\boldsymbol{x})$ or $f_5(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ is changed into $\widetilde{R}_3$ in order to enlarge the difference between them. Nevertheless there is still the inferior solution, e.g. the satisfying degrees of all the objectives as $(0.9684, 0.9684, 0.55869, 0.7700, 0.9684)$ when $\lambda = 0.8$. Certainly, their paper has indicated that this case may exist and give the further discussion. Accordingly, the more decision of determining the proper fuzzy binary relation has to be done. All of the aboves imply that our method is more efficient and simple.

5.3. Flexibility

The flexibility of our method is shown by means of altering the extent of relative importance of the objectives but remaining the original linguistic terms order. The diverse preference extent and same order makes the analyst preserve the identical model (37). Therefore he or she only selects the proper optimization result for enlarged extent of relative importance. For example, DM determines the preferred solution with $\gamma \in (-0.0980, -0.1267)$ for the old importance extent. While the new one is: $f_1(\boldsymbol{x})$ and $f_5(\boldsymbol{x})$ are "very important"; $f_2(\boldsymbol{x})$ is "important"; $f_4(\boldsymbol{x})$ is "unimportant"; $f_3(\boldsymbol{x})$ is "very unimportant". Then the solution satisfies DM when $\gamma \in (-0.2000, -0.3333)$. It is obvious the new γ is more suitable to this new preference extent.

5.4. Sensitivity

The sensitivity of our method can be demonstrated by solving the problem under different linguistic terms of DM. Here, the preference requirement becomes: $f_1(\boldsymbol{x})$ and $f_5(\boldsymbol{x})$ are "very important"; $f_3(\boldsymbol{x})$ is "somewhat important"; $f_4(\boldsymbol{x})$ is "important", and $f_2(\boldsymbol{x})$ is "general". Then the comparison inequities in optimization model (37) will, respectively, be rewritten as follows:

$$\begin{cases} \mu^*_{f_3} - \mu^*_{f_1} \leqslant \gamma, \\ \mu^*_{f_3} - \mu^*_{f_5} \leqslant \gamma, \\ \mu^*_{f_4} - \mu^*_{f_3} \leqslant \gamma, \\ \mu^*_{f_2} - \mu^*_{f_4} \leqslant \gamma. \end{cases} \tag{39}$$

The optimization results are shown in the following Table 4.

In the above case, the order of importance among $f_2(\boldsymbol{x}), f_3(\boldsymbol{x})$ and $f_4(\boldsymbol{x})$ are changed. When $\lambda = 0.8$, i.e. $\gamma = -0.0925$, the result is satisfying to DM.

Example 2 (*Li et al., 2004; Pal and Moitra, 2003*). Find $x(x_1,x_2)$ to satisfy

$$\begin{cases} f_1(\boldsymbol{x}): & 5x_1 + 2x_1^2 \gtrsim 55, \\ f_2(\boldsymbol{x}): & 4x_2 + 3x_2^2 \cong 41, \\ f_3(\boldsymbol{x}): & 7.5x_2 + x_2^3 \lesssim 45, \\ f_4(\boldsymbol{x}): & x_1 \gtrsim 4, \\ f_5(\boldsymbol{x}): & x_2 \lesssim 3. \end{cases} \tag{40}$$

Subject to

Table 3
Optimization results by Aköz and Petrovic's method for Example 1

λ	Sum of desirable satisfying degrees	Sum of the satisfying degrees of fuzzy importance	Individual satisfying degree	Solution
(0.52–1)	4.3279	1.9333		
		(0.9813, 1.0000, 0.6050, 0.7750, 0.9667)	(0.0000, 9.7500, 0.0000, 15.8750)	
(0.4–0.51)	4.314	1.9481		
		(0.9740, 1.0000, 0.5883, 0.7776, 0.9740)	(0.0649, 9.7403, 0.0000, 15.7790)	
(0.25–0.39)	3.9164	2.2060		
		(0.9461, 0.8431, 0.4412, 0.7400, 0.9461)	(1.4801, 8.1155, 0.0000, 13.9270)	
(0.21–0.24)	3.5682	2.3182		
		(1.0000, 0.84091, 0.0455, 0.6818, 1.0000)	(1.3636, 8.6364, 0.0000, 12.2730)	
(0.14–0.2)	3.4954	2.3364		
		(1.0000, 0.8318, 0.0000, 0.6636, 1.0000)	(1.3475, 8.6018, 0.0507, 12.0010)	
(0–0.13)	2.3498	2.5222		
		(0.7833, 0.5222, 0.0000, 0.2611, 0.7833)	(1.2515, 5.3000, 1.8232, 9.1433)	

Table 4
Optimization results of Example 1 for sensitivity

λ	Sum of desirable satisfying degrees	Importance difference	Individual desirable satisfying degree	Individual satisfying degree	Solution
0.15	4.3193	0.2122	(0.9637, 1.0000, 0.5756, 0.7878, 0.9922)	(0.9637, 1.0000, 0.5756, 0.7878, 0.9922)	(0.0000, 9.9419, 0.0000, 15.8430)
0.3	3.9752	−0.0742	(1.0000, 0.6325, 0.7809, 0.7067, 0.8551)	(1.0000, 0.9049, 0.7809, 0.7067, 0.8551)	(0.0000, 8.7254, 0.1879, 16.0458)
0.8	3.9147	−0.0925	(0.8939, 0.6165, 0.8014, 0.7090, 0.8939)	(0.8939, 0.9344, 0.8014, 0.7090, 0.8939)	(0.0000, 8.7593, 0.4453, 16.0401)
1.0	2.8743	−0.3194	(0.9581, 0.0000, 0.6387, 0.3194, 0.9581)	(0.9581, 0.9918, 0.6387, 0.7644, 0.9581)	(0.0000, 9.5915, 0.0942, 15.9014)

$$\begin{cases} x_1 + 2x_2 \leqslant 10, \\ x_1 \leqslant 10, \\ x_2 \leqslant 5, \\ x_1, x_2 \geqslant 0, \end{cases} \tag{41}$$

where there is no constraint "x_1,x_2 are integers", and the priority requirement is replaced by the importance requirement. The tolerant ranges of these objectives are (35,55), (33,49), (45,60), (2,4) and (3,4), respectively. The fuzzy importance requirement is: $f_1(x)$ is "very important"; $f_2(x)$ is "somewhat important"; $f_3(x)$ is "important"; $f_4(x)$ and $f_5(x)$ is "general".

According to the fuzzy relations include inequalities and equalities, the reformulation is given as

$$\begin{cases} \max & \sum_{i=1}^{5} \mu^*_{f_i}/5 - \lambda \cdot \gamma \\ \text{s.t.} & 5x_1 + 2x_1^2 + n_1 - p_1 = 55, \\ & 4x_2 + 3x_2^2 + n_2 - p_2 = 41, \\ & 7.5x_2 + x_2^3 + n_3 - p_3 = 45, \\ & x_1 + n_4 - p_4 = 4, \\ & x_2 + n_5 - p_5 = 3, \\ & 1 - n_1/20 \geqslant \mu^*_{f_1}, \\ & 1 - n_2/8 - p_2/8 \geqslant \mu^*_{f_2}, \\ & 1 - p_3/15 \geqslant \mu^*_{f_3}, \\ & 1 - n_4/2 \geqslant \mu^*_{f_4}, \\ & 1 - p_5/1 \geqslant \mu^*_{f_5}, \\ & \mu^*_{f_2} - \mu^*_{f_1} \leqslant \gamma, \\ & \mu^*_{f_3} - \mu^*_{f_2} \leqslant \gamma, \\ & \mu^*_{f_4} - \mu^*_{f_3} \leqslant \gamma, \\ & \mu^*_{f_5} - \mu^*_{f_3} \leqslant \gamma, \\ & n_1 \leqslant 20, \quad n_2, p_2 \leqslant 8, \quad p_3 \leqslant 15, \quad n_4 \leqslant 2, \quad p_5 \leqslant 1, \\ & n_i, p_i \geqslant 0, \quad \mu^*_{f_i} \geqslant 0, \quad n_i \cdot p_i = 0, \quad i = 1, \ldots, 5, \\ & \text{system constraints (41)} \end{cases} \tag{42}$$

The optimization results for different λ are written in Table 5.

when $\lambda = 0.8$, i.e. $\gamma = -0.0755$, the optimization result is satisfying for DM. If DM requires the higher preference extent, the solution for $\gamma = -0.3146$ will be proper. This demonstrates the flexibility.

For computing minimum λ^*, solve the programming model (42) with the optimization objective "$-\gamma$" and acquire the corresponding solution:

$$\begin{cases} \boldsymbol{x}^* = (4.0885, 2.9558), \\ \boldsymbol{n}^* = (1.1268, 2.9671, 0.0000, 0.0000, 0.0442), \\ \boldsymbol{p}^* = (0.0000, 0.0000, 2.9917, 0.0885, 0.0000), \\ \gamma^* = -0.3146. \end{cases}$$

According to the above solution, the auxiliary linear programming model is given as

$$\begin{cases} \max & \sum_{i=1}^{5} \mu^*_{f_i}/5 - \lambda \cdot \gamma \\ \text{s.t.} & 1 - n_1^*/20 \geqslant \mu^*_{f_1}, \\ & 1 - n_2^*/8 - p_2^*/8 \geqslant \mu^*_{f_2}, \\ & 1 - p_3^*/15 \geqslant \mu^*_{f_3}, \\ & 1 - n_4^*/2 \geqslant \mu^*_{f_4}, \\ & 1 - p_5^*/1 \geqslant \mu^*_{f_5}, \\ & \mu^*_{f_2} - \mu^*_{f_1} \leqslant \gamma, \\ & \mu^*_{f_3} - \mu^*_{f_2} \leqslant \gamma, \\ & \mu^*_{f_4} - \mu^*_{f_3} \leqslant \gamma, \\ & \mu^*_{f_5} - \mu^*_{f_3} \leqslant \gamma, \\ & \gamma \leqslant 0. \end{cases} \tag{43}$$

when $\lambda = 0$, (43) is also solved by simplex method. After the simplex conversion, we can see that the solution keeps same when $\lambda > 1$. Then λ^o is equal to 1. Substituting $\lambda = 1$ into (42), and obtain the result:

$$\mu_1^o = 0.8394, \quad \mu_2^o = 0.7639, \quad \mu_3^o = 0.6884, \quad \mu_4^o = 0.6129, \quad \mu_5^o = 0.6129, \quad \gamma^o = -0.0755.$$

where $\gamma^o \neq \gamma^*$. Obviously λ^o is not final result. The following equality is formulated.

$$\mu^o/k - \lambda \cdot \gamma^o = \mu^*/k - \lambda \cdot \gamma^*,$$

where $\mu^o = 3.5174$, $\gamma^o = -0.0755$, $\mu^* = 1.8874$, $\gamma^* = -0.3146$. Then $\lambda = 1.37$. Substituting λ into (42), γ is equal to -0.3146. Then *1.37* is the minimum λ^*.

Similarly, the different fuzzy importance of DM can show the sensitivity of this proposed approach. The preference requirement becomes: $f_4(\boldsymbol{x})$ and $f_5(\boldsymbol{x})$ is "very important"; $f_2(\boldsymbol{x})$ is "somewhat

Table 5
Optimization results with different λ for Example 2

λ	Sum of desirable satisfying degrees	Importance difference	Individual desirable satisfying degree	Individual satisfying degree	Solution
0.05	4.4172	0.3011	(1.0000, 0.5580, 0.8592, 1.0000, 1.0000)	(1.0000, 0.5580, 0.8592, 1.0000, 1.0000)	(4.1410, 2.9295)
0.3	4.3799	0.1907	(0.9520, 0.6186, 0.8093, 1.0000, 1.0000)	(0.9520, 0.6186, 0.8093, 1.0000, 1.0000)	(4.0962, 2.9519)
0.8	3.5175	−0.0755	(0.8394, 0.7639, 0.6884, 0.6129, 0.6129)	(0.8395, 0.7639, 0.6885, 0.9950, 0.9950)	(3.9900, 3.0050)
1.5	1.8874	−0.3146	(0.9437, 0.6291, 0.3146, 0.0000, 0.0000)	(0.9437, 0.6291, 0.8005, 1.0000, 1.0000)	(4.0885, 2.9558)

Table 6
Optimization results of Example 2 for sensitivity

λ	Sum of desirable satisfying degrees	Importance difference	Individual desirable satisfying degree	Individual satisfying degree	Solution
0.2	4.3640	0.1438	(0.9318, 0.6443, 0.7880, 1.0000, 1.0000)	(0.9318, 0.6443, 0.7880, 1.0000, 1.0000)	(4.0773, 2.9613)
0.8	3.8976	−0.1649	(0.4827, 0.8125, 0.6476, 0.9774, 0.9774)	(0.8027, 0.8125, 0.6477, 0.9774, 0.9774)	(3.9548, 3.0226)
1.0	3.5000	−0.2500	(0.2500, 0.7500, 0.5000, 1.0000, 1.0000)	(0.8500, 0.7500, 0.7000, 1.0000, 1.0000)	(4.0000, 3.0000)
1.5	3.0000	−0.3333	(0.0000, 0.6667, 0.3333, 1.0000, 1.0000)	(0.9318, 0.6667, 0.7694, 1.0000, 1.0000)	(4.0598, 2.9696)

DM can select $\gamma = -0.1649$ as his satisfying solution.

important"; $f_3(\boldsymbol{x})$ is "important"; $f_1(\boldsymbol{x})$ is "general". Then ranking inequities in optimization model (42) will, respectively, be altered. The optimization results are shown in the following Table 6.

Remark 1. From the above tables, it is seen that the sum of desirable satisfying degrees decreases and the importance difference between the objectives enhances with the increment of parameter λ. The different results can satisfy the different types of importance requirement of DM.

Remark 2. According to Chen and Tsai (2001), if DM wants to distinguish relative importance between different objectives, he must give the corresponding aspiration satisfying degree for each objective. However, it is difficult to do so for DM who has only imprecise information in real life, even there may be no solution with the given values.

Remark 3. The results are acquired by regulating parameter λ. Then DM can find the satisfying one from the alternatives according to the importance extent of the objectives. The various desirable satisfying degrees may result in the identical solution.

6. Conclusions

Based on Chen and Tsai (2001) and Li et al. (2004), this paper presents the satisfying optimization method based on goal programming for fuzzy multiple objective optimization problem. This method realizes the trade-off between optimization and fuzzy importance requirement. DM can find the appropriate alternative according to his intention from various solutions by regulating parameter λ. The results of the examples show its efficiency, flexibility and sensitivity for the optimization problems with three types of fuzzy relations. It can be used in many real-world DM problems. Further how to solve the non-convex and uncertain stochastic optimization problem will be discussed in depth in a forthcoming paper.

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